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Abelian subalgebras on Lie algebras

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Abelian subalgebras play an important role in the study of Lie algebras and their properties and structures. In this paper, the historical evolution of this concept is shown, analyzing the current status for the research on this topic. So, the main results obtained from previous years are indicated and commented here. Additionally, a list of some related open problems is also given.

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1. Introduction

Apart from the study of Lie Theory starting from a purely theoretical point of view, extensive research about this theory exists due to its applications in Engineering, Physics and, above all, Applied Mathematics. However, some aspects of Lie algebras have to be still studied. Indeed, the classification of nilpotent and solvable Lie algebras is still an open problem, although the classification of other types of Lie algebras (like semisimple and simple ones) were already obtained in 1890. In order to solve these and other problems, the need of studying other properties of Lie algebras arises. Considering abelian Lie subalgebras constitutes the main goal of this survey, in which we have shown the historical evolution in the study of this concept, analyzing the current status of the research on this topic.

It is also convenient to indicate which motivations lead to obtain more information about Lie algebras in general. In a high percentage, the reasons lie in the possibility of using Lie algebras and their properties as tools in the study of several topics in Physics and Economics, for instance. Indeed, at present, Lie algebras and groups are widely used in Modern Physics. For example, a classic use of Lie Theory corresponds with the study of symmetries (see [30, 52]). Nowadays, symmetries are not limited to those geometrical versions of space–time; but there are other new symmetries associated with "internal" degrees of freedom of particles and fields. A more interesting tool is given by the contractions of Lie algebras. Let us recall that the invariant $\mathcal{M}(\mathfrak{g})$, defined as the maximal dimension of the abelian subalgebras of $\mathfrak{g}$, is the same under contractions, which is a very important physical application. Regarding to possible economic and finance applications, we advise to consult [29] as starting point.

The structure of this survey is the following: Section 2 recalls those more general concepts on Lie algebras which will be cited throughout this survey. Section 3 constitutes a historical introduction, including the earliest papers dealing with the original problem stated by Schur [58] and Jacobson [32] about the computation of abelian subalgebras of a given Lie algebra, together with the answers given by several researchers. Besides, we comment most of the recent papers dealing particularly with abelian subalgebras of Lie algebras. Section 4 is devoted to show a particular aspect of this research line: the maximal abelian dimension of a given Lie algebra (i.e. the maximum among the dimension of the abelian subalgebras) and its computation. Let us note that this research has been carried out for the last five years by many different authors (including the ones of this survey). In Sec. 5, we show all the research obtained by the authors of this survey on the topic of abelian subalgebras of a Lie algebra. Finally, the last section draws some conclusions about this research line, proposing some future works.

2. Preliminaries

This section is devoted to recall some preliminary concepts and results on Lie algebras. For a general overview, the interested reader can consult [67]. From here on and unless the contrary is stated, only finite-dimensional Lie algebras over the complex number field are considered.

A *Lie algebra* $\mathfrak{g}$ over an arbitrary field $\mathbb{K}$ is a vector space over $\mathbb{K}$ endowed with a second inner law, named the *bracket product*, and verifying the following three properties:

(1) Bilinearity: $[\alpha u_1 + \beta u_2, v] = \alpha[u_1, v] + \beta[u_2, v]$, $\forall\, \alpha, \beta \in \mathbb{K}, \forall\, u_1, u_2, v \in \mathfrak{g}$ and $[u, \alpha v_1 + \beta v_2] = \alpha[u, v_1] + \beta[u, v_2]$, $\forall\, \alpha, \beta \in \mathbb{K}, \forall\, u, v_1, v_2 \in \mathfrak{g}$.
(2) Commutativity: $[u, u] = 0$, $\forall\, u \in \mathfrak{g}$.
(3) Jacobi identity: $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$, $\forall\, u, v, w \in \mathfrak{g}$.

Let us consider a Lie algebra $\mathfrak{g}$ with a basis $\{e_1, \ldots, e_n\}$. This basis can be characterized by the *structure constants* (or *Maurer–Cartan constants*), defined by $[e_i, e_j] = \sum c_{i,j}^h e_h$ for any $1 \leq i, j \leq n$. These constants determine the whole algebra.

The second condition in the previous definition, together with the bilinearity of the bracket product, implies the skew-symmetry property

$$[u, v] = -[v, u] \quad \forall\, u, v \in \mathfrak{g}.$$

Given a Lie algebra $\mathfrak{g}$, a vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *subalgebra* if the following condition holds

$$[u, v] \in \mathfrak{h}, \quad \forall\, u, v \in \mathfrak{h}.$$

Moreover, we will say that the subalgebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ if $\mathfrak{h}$ satisfies the condition

$$[h_1, g_1] \in \mathfrak{h}, \quad \forall\, h_1 \in \mathfrak{h}, \ \forall\, g_1 \in \mathfrak{g}.$$

There exist three different types of Lie algebras: solvable algebras, semisimple ones and those which do not belong to these two previous types, but can be expressed as a semidirect sum of two algebras of the previous types.

It is said that a Lie algebra $\mathfrak{g}$ over a field of characteristic zero is *semisimple* if $\mathfrak{g}$ does not contain any non-zero proper abelian ideal. Additionally, the Lie algebra $\mathfrak{g}$ is said to be *simple* if it is not abelian and does not contain any non-zero proper ideal.

Given a finite-dimensional Lie algebra $\mathfrak{g}$, its *upper central series* is defined by

$$\mathcal{C}_1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}_2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}],\ \mathcal{C}_3(\mathfrak{g}) = [\mathcal{C}_2(\mathfrak{g}), \mathcal{C}_2(\mathfrak{g})], \ldots, \mathcal{C}_k(\mathfrak{g}) = [\mathcal{C}_{k-1}(\mathfrak{g}), \mathcal{C}_{k-1}(\mathfrak{g})], \ldots.$$

If there exists $m \in \mathbb{N}$ such that $\mathcal{C}_m(\mathfrak{g}) \equiv 0$, the Lie algebra $\mathfrak{g}$ is called *solvable*.

Analogously, the *lower central series* of $\mathfrak{g}$ is defined by

$$\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}^2(\mathfrak{g}) = [\mathcal{C}^1(\mathfrak{g}), \mathfrak{g}], \mathcal{C}^3(\mathfrak{g}) = [\mathcal{C}^2(\mathfrak{g}), \mathfrak{g}], \ldots, \mathcal{C}^k(\mathfrak{g}) = [\mathcal{C}^{k-1}(\mathfrak{g}), \mathfrak{g}], \ldots.$$

If there exists $m \in \mathbb{N}$ such that $\mathcal{C}^m(\mathfrak{g}) \equiv 0$, the Lie algebra $\mathfrak{g}$ is called *nilpotent*. It is easy to see that every nilpotent Lie algebra is also solvable.

A special class of solvable Lie algebras is formed by abelian algebras. A Lie algebra $\mathfrak{g}$ is *abelian* if $[v, w] = 0$, for all v, $w \in \mathfrak{g}$.

A very important and interesting abelian subalgebra (in fact, an abelian ideal) in a given Lie algebra $\mathfrak{g}$ is the *center* of the algebra, which is defined as

$$\operatorname{cen}(\mathfrak{g}) = \{u \in \mathfrak{g} \mid [u, v] = 0, \forall\, v \in \mathfrak{g}\}.$$

Other useful ideals of $\mathfrak{g}$ are its *radical* (i.e. the sum of all its solvable ideals) and its *nilradical* (i.e. the sum of all its nilpotent ideals). It is immediate that the nilradical is included in the radical for any given Lie algebra.

Finally, the *maximal abelian dimension* of $\mathfrak{g}$, which will be denoted by $\mathcal{M}(\mathfrak{g})$, is defined as the maximum among the dimensions of its abelian subalgebras.

3. Historical Evolution

In this section, we show the historical evolution followed in the papers dealing with abelian subalgebras of a given Lie algebra. For many years, a lot of properties have been studied for abelian subalgebras and abelian ideals. In fact, the problem of computing the maximal abelian subalgebra contained in a Lie algebra is very old.

The first author dealing with this topic was Schur [58], who studied in 1905 the dimension of the maximal abelian subalgebras contained in the Lie algebra of $n \times n$ square matrices. Schur proved that *the maximum number of linearly independent commuting* $n \times n$ *matrices over a algebraically closed field is* $[\frac{n^2}{4}]+1$. Let us note that this result was obtained only over an algebraically closed field such as the complex number field. Almost forty years later, in 1944, Jacobson [32] gave a simpler proof of Schurs results, extending them from algebraically closed fields to arbitrary fields. This fact allowed several authors to deal later with the study of the maximal abelian dimension of many different types of Lie algebras.

As it is well known, Lie algebras can be distinguished in three different types: solvable ones, semisimple ones and the remaining algebras which do not belong to any of previous types. More concretely, Levi [43] and Malcev [45] proved (in 1905 and 1945, respectively) that every finite-dimensional Lie algebra can be expressed as a semidirect sum of a semisimple subalgebra and its radical (i.e. a solvable ideal). Hence, classifying all Lie algebras can be reduced to obtain the classification of both semisimple and solvable Lie algebras.

Apart from that, it is also well known that every semisimple Lie algebra can be decomposed in a direct sum of simple Lie algebras. By taking advantage of the classification given by Killing [33–36] and Cartan [7] (among others) for simple Lie algebras at the end of the nineteenth century, Malcev [45] could compute the abelian subalgebras of maximum dimension of semisimple Lie algebras in 1945.

Later, Kostant [38] gave in 1965 a link between Malcev's result and the maximal eigenvalue of the Laplacian acting on the exterior powers of the adjoint representation. In 1998, Kostant himself [39] also dealt with the topic of abelian ideals in a Borel subalgebra of a given Lie algebra $\mathfrak{g}$ by studying the proof of the following theorem, which has been traditionally attributed to Peterson in that article: the number of abelian ideals in a fixed Borel subalgebra of $\mathfrak{g}$ is 2^r, where $r = \text{rank}(\mathfrak{g})$. Regarding this, some authors like Cellini and Papi [16, 17] have recently obtained new properties of the maximal abelian ideals in a Borel subalgebra as well as a generalization of Peterson's theorem from abelian ideals to ad-nilpotent ones.

Moreover, the authors of the present survey also studied the maximum among the dimensions of abelian subalgebras of Heisenberg algebras in [51], and of the Lie algebras $\mathfrak{g}_n$ and $\mathfrak{h}_n$ (formed of strictly upper-triangular matrices and upper-triangular matrices, respectively) in [3, 4, 8]. Besides, an algorithmic method was also obtained in [12] to compute abelian subalgebras of a Lie algebra whose dimension is as greater as possible.

4. Abelian Subalgebras of a Lie Algebra

The problem of classifying abelian subalgebras of Lie algebras has a long history, starting from the Killing–Cartan classification of semisimple Lie algebras. However, the problem of determining such subalgebras is far from being solved, due to the lack of structural criteria for general types of Lie algebras.

Bearing in mind the results obtained by Schur [58], Jacobson [32] and Malcev [45], some authors have dealt with this topic in order to achieve new results on this matter of abelian subalgebras of an arbitrary Lie algebra. For example, let us recall that Suprunenko and Tyshkevich [60] dealt in 1968 with the problem of determining maximal abelian subalgebras of nilpotent type of a Lie algebra. Therefore, in this section, we are going to analyze and summarize some important papers about this subject.

First, Bratzlavsky [6] studied in 1974 the law of some nilpotent Lie algebras of dimension n and class $n-1$. These algebras are known as filiform Lie algebras and were introduced by Vergne [69]. More concretely, Bratzlavsky obtained the canonical forms for the structure of filiform Lie algebras having an abelian derived Lie algebra. One of the results obtained in that article was the following theorem.

Theorem 4.1. *For a filiform Lie algebra whose derived Lie algebra is abelian, there exists a basis* $\{x_1, \dots, x_n\}$ *such that*

$$[x_1, x_i] = x_{i+1}, 2 \leq i \leq n-1; \quad [x_i, x_j] = 0, \quad \textit{for } 3 \leq i < j \quad \textit{and}$$
$$[x_2, x_i] = \sum \lambda_r x_{i+2+r}, \quad \textit{for } 3 \leq i \leq n-2, \quad \textit{and } r \leq n-i-2.$$

Later, Amayo and Stewart [1] in 1974 wondered whether there would be some inclusions between the following families of Lie algebras: (a) those containing finite-dimensional abelian ideals; (b) those containing finite-dimensional nilpotent ideals; (c) those satisfying the maximal condition for abelian, nilpotent and solvable ideals respectively; and finally, (d) those satisfying the minimal condition for abelian, nilpotent and solvable ideals respectively.

Answering that question, Kubo [41] obtained in 1978 the following two results by using different tools like tensorial extensions, adjoint transformations and central simple Lie algebras, for instance.

Theorem 4.2. *The class of Lie algebras which contains finite-dimensional abelian ideals is not equal to the one which contains finite-dimensional nilpotent ideals.*

Theorem 4.3. *It is possible to find a Lie algebra verifying the maximal and minimal condition for abelian ideals which does not verify the same conditions for solvable ideals.*

Three years later, Zaicev [70] in 1981 proved a result about the relative distribution of an abelian ideal and a positive polarization in an arbitrary Lie algebra, applying this result to find representations of Lie groups with an abelian normal subgroup.

In that same article, Zaicev developed a theory (introduced by Kirillov [37]) about the extension of the orbit to solvable Lie groups. He used the notion of polarization to deal with arbitrary Lie algebras and Lie groups in general instead of solvable ones. Moreover, Zaicev considered abelian ideals instead of abelian subalgebras, which requires more restrictive conditions. Let us note that this notion of polarization has also independent algebraic interest.

In his work, Zaicev used the following main tools: real Lie groups, their associated Lie algebras, dual spaces of Lie algebras, stationary groups, the total positive polarization of an element in the dual space of a given Lie algebra with respect to a stationary group and a regular intersection with a polarization.

Some authors thought introducing new models of Lie algebras was completely necessary, since the classification of Lie algebras was an unsolved problem. For example, Bowman and Towers [5] studied in 1996 those Lie algebras whose proper subalgebras are nilpotent-by-abelian but which themselves are not nilpotent-by-abelian. They analyzed the structure and existence of these algebras. Previously, other authors (like Elduque [19], Farnsteiner [22, 23], Gein [26, 27] and Varea [68], for instance) had studied simple semiabelian Lie algebras. Besides this, *almost nilpotent* Lie algebras (i.e. those containing a finite-dimensional nilpotent ideal) were studied and analyzed by Stitzinger [59], Gein and Kuznecov [28], Towers [63, 65] and Farnsteiner [21, 24], and *almost supersolvable* Lie algebras by Towers [64] and Elduque and Varea [20].

Bowman and Towers [5] obtained several results when studying almost nilpotent-by-abelian Lie algebras. Some of them are the following.

Lemma 4.1. *Let $\mathfrak{L}$ be any Lie algebra. Then $\mathfrak{L}$ is nilpotent-by-abelian if and only if its derived algebra $\mathcal{C}^2(\mathfrak{L}) = \mathfrak{L}^2$ is nilpotent.*

Theorem 4.4. *Let $\mathfrak{L}$ be any Lie algebra over a field $\mathbb{F}$ of characteristic zero. Then the following statements are equivalent*:

- *$\mathfrak{L}$ is almost nilpotent-by-abelian.*
- *$\mathfrak{L}$ is simple semiabelian or else $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{F})$.*

To obtain these both results, the main tools used were Frattini subalgebras and ideals, algebraically closed fields and the following structures of Lie algebras: Heisenberg, solvable, nilpotent, semisimple and simple. Let us note that Frattini structures have relation with maximal subalgebras and maximal ideals. In their article, Bowman and Towers also considered the cases of an algebraically closed field of characteristic zero and one of characteristic $p > 0$. In this sense, some theorems about the structure of certain solvable almost nilpotent-by-abelian algebras were set.

Additionally, some articles can also be found dealing with important and useful properties of Lie subalgebras like decomposability. For example, Petravchuk [56] proved in 1999 that a Lie algebra $\mathfrak{L}$ over an arbitrary field can be decomposed into the sum $\mathfrak{L} = \mathfrak{A} + \mathfrak{B}$ of an almost abelian subalgebra $\mathfrak{A}$ and a finite-dimensional subalgebra $\mathfrak{B}$ over its center. His main goal was to prove that this algebra is almost solvable (i.e. containing a solvable ideal of finite dimension) and that the sum of an abelian Lie algebra and an almost abelian one is an almost solvable Lie algebra.

To do this, he firstly developed an historical introduction of the problem, recalling one of Ito's classic theorems (see [31, 18], for instance) about the solvability of a product of two abelian groups. This result is also true for Lie algebras: *A Lie algebra decomposable into the sum of two of its Abelian subalgebras is solvable.* Moreover, Petravchuk himself recalled the following open problem: *it is unknown whether the product of two almost abelian groups is solvable.* In this sense, the main result in [56] pursued to answer this open problem. Its statement is the following.

Theorem 4.5. *Let $\mathfrak{L}$ be a Lie algebra over an arbitrary field that is decomposable into the sum $\mathfrak{L} = \mathfrak{A} + \mathfrak{B}$ of a finite-dimensional subalgebra $\mathfrak{A}$ over its center and an almost abelian subalgebra $\mathfrak{B}$. Then, the algebra $\mathfrak{L}$ is almost solvable.*

As a consequence, the following result holds.

Corollary 4.1. *The sum of abelian and almost abelian Lie groups is almost solvable.*

To prove Theorem 4.5, some local results were considered in the article. Some of them were necessary conditions for almost solvable Lie algebras and sufficient conditions for almost solvable Lie algebras or Lie algebras containing a solvable or almost solvable subalgebra. Besides, some statements about operations with brackets and sums were also proved.

Later, some papers appeared in about 2004 dealing with abelian ideals in Borel subalgebras. As a sake of example, Suter [61] started from a complex Lie algebra $\mathfrak{g}$ and a fixed Borel subalgebra $\mathfrak{b}$ of it, describing all the abelian ideals of $\mathfrak{b}$ in a uniform way and independently of the classification of complex simple Lie algebras. Besides, as an application of this description, a formula was obtained for the maximal dimension of an abelian subalgebra of $\mathfrak{g}$. In this paper, the maximal dimension among the abelian subalgebras of $\mathfrak{g}$ was determined purely in terms of certain invariants, such as the dual Coxeter number of $\mathfrak{g}$ and the number of positive roots of some associated root subsystems of $\mathfrak{g}$. Other tools used in this article were fundamental alcoves (see [2, p. 70]), symmetric groups, Dihedral and Weyl groups and the Hasse graph of Young's lattice.

In this way, Suter answered and solved Panyushev and Röhrle's question [55] about a uniform explanation for the one-to-one correspondence between maximal abelian ideals in the Borel subalgebra $\mathfrak{b}$ and long simple roots. This answer was given by emerging all positive long roots in a natural way (a very interesting overview about this question can be consulted in the very recent paper [44]). Finally, Suter also gave a generalization of the symmetry property of a certain subset of Young's lattice (namely, the lattice of integer partitions).

Simultaneously, Cellini, Frajria and Papi [15] studied some properties of abelian subalgebras in the particular case of $\mathbb{Z}_2$-graded Lie algebras. More concretely, they considered a simple $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$ and a fixed Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$. The main goal of [15] was to describe and enumerate abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$, which is a problem previously posed by Panyushev [54]. Besides, some formulas were obtained in terms of combinatorial data associated with the $\mathbb{Z}_2$-graduation.

The main interest of this question lies in Kostant's theorem [38] relating abelian subalgebras to the maximal eigenvalue of the Casimir element. This theorem was later generalized to $\mathbb{Z}_2$ by Panyushev [53], who solved the problem posed in [54] for the very special case of the little adjoint module by identifying abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$ with abelian ideals of long roots of a Borel subalgebra of the Langlands dual of $\mathfrak{g}_0$.

The approach given in [15] for describing abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$ is based on a suitable combination of some ideas from Garland and Lepowsky [25] and Kostant [39, 40], using several types of Lie algebras such as affine Kac–Moody Lie algebras, Cartan algebras, graded algebras and Borel algebras.

Later, a recent new research was developed in 2008 by Romanovskii and Shestakov [57], who proved some properties of abelian Lie algebras. More concretely, they studied whether a wreath product of abelian Lie algebras is Noetherian with respect to the equations of the universal enveloping algebra. This was done with the objective of obtaining some improvements about algebraic geometry over Lie algebras, by proving that a wreath product of two abelian finite-dimensional Lie algebras over a field of characteristic zero is Noetherian with respect to the equations of the universal enveloping algebra. They previously recalled several earlier papers constructing examples of groups that are not equationally Noetherian in several cases, as well as similar examples for Lie algebras over an arbitrary field.

By using different techniques (like the concepts and properties of equationally Noetherian Lie algebras, abelian normal subgroups, Noetherian Lie algebras, soluble free Lie algebras, algebraic subsets or coordinate algebras), the main result obtained was the following theorem.

Theorem 4.6. *A wreath product of two abelian finite-dimensional Lie algebras over a field of characteristic* 0 *is Noetherian with respect to the equations of a universal enveloping algebra.*

This implies that every soluble free Lie algebra of index 2 and finite rank has this property. So, Romanovskii and Shestakov also proved the following corollary.

Corollary 4.2. *An index* 2 *soluble free Lie algebra of finite rank is Noetherian with respect to the equations of a universal enveloping algebra.*

For advancing in this subject, it was very useful to deal with the dimension of the abelian subalgebras of a given Lie algebra. In this sense, Milentyeva [46] studied this topic, obtaining some functions as bounds for the dimension of an abelian subalgebra in finite-dimensional associative algebras and Lie algebras. The same results were also obtained for the largest abelian subgroup of a Lie group.

Moreover, the growth of the functions previously developed to bound these dimensions was studied in [46] too. To do so, several different cases were considered depending on the field which the algebras are defined over. In this way, the functions were well defined and finite for the case of complex and real number fields, having quadratic growth in other cases.

The families of associative and Lie algebras satisfying that the dimension of all its abelian subalgebras is at most n, denoted by condition $A(n)$, were also studied by Milentyeva in [46]. Besides, the set of the greatest integer h such that there exists a Lie algebra (respectively, an associative algebra) of dimension h and verifying the condition $A(n)$ was considered. In addition, the goal was to find the greatest k satisfying that there exists a Lie group of dimension k over an arbitrary field verifying that the dimension of all its abelian Lie subgroups is less than or equal to n.

Because of the great importance of Milentyeva's article, we think appropriate describing more completely the procedure. The general structure of the article is the following: first, an introduction showed the most important and essential concepts and definitions. In this section, the author expounded the main result of the article, related to inequalities for the bound functions. Next, a section is devoted to study quadratic upper bounds for the relation between the dimension of a given Lie algebra and the dimension of its maximal abelian subalgebra. All this study was carried out for Lie algebras over both the complex and real number fields, as well as repeating this study for associative algebras. The following section computed quadratic lower bounds for Lie algebras over an arbitrary field. Finally, the author showed more details about the bounds given for nilpotent algebras and groups.

One year later, Milentyeva [47] continued his work computing the functions which bound the dimensions of finite-dimensional nilpotent both associative and Lie algebras of class 2 over an algebraically closed field in terms of the dimensions of their abelian subalgebras. Whereas her previous paper only gave bounds for these functions, they were now completely determined and computed as a expression of a value n which bounds (or is exactly equal to) the maximal abelian dimension of the associative or Lie algebra given for these functions.

In this way, the main theorem of [47] consisted of the mathematical expression of these functions written with respect to the value n. To obtain this result, Milentyeva had to prove a main lemma which assured the existence of a vector subspace in a fixed and given vector space and which is simultaneously isotropic for all of the components in any tuple (with a fixed dimension) of alternating bilinear forms. The proof of this lemma was based on the application of Zariski Topology over projective spaces and the notions of projective and quasiprojective varieties in those spaces, as well as the notion of regular map from quasiprojective varieties to projective spaces (not necessarily the same containing such varieties). Other mathematical objects used in the proof are Grassman varieties, Plcker coordinates of a vector subspace and Schubert cells in a Grassman variety.

The last stage of Milentyeva's theorem was to reduce the associative case to the Lie case. In fact, she proved that the value of the functions was the same independently of using associative algebras or Lie algebras. After this, she determined both upper and lower bounds of the function for the Lie case, by using the main lemma given in this article and the results already obtained in [46]. The proof concluded when she obtained that upper and lower bounds were equal to each other.

5. The Concept of Maximal Abelian Dimension of a Lie Algebra

As we have already commented in Sec. 1, for the last five years, several authors have dealt with the general topic of (abelian) subalgebras of a Lie algebra: more particularly, the maximal abelian dimension of a Lie algebra $\mathfrak{g}$, that is, the maximum among the dimension of its abelian subalgebras and denoted by $\mathcal{M}(\mathfrak{g})$.

As it has already been commented, Schur [58] was the first author dealing with this concept, although indirectly a century ago. Later, Malcev [45] obtained in 1945 the maximal abelian dimension of simple Lie algebras. In the following table, the second row contains the dimensions of the corresponding Lie algebras and their maximal abelian dimensions are shown in the third row

Type of $\mathfrak{g}$	E_6	E_7	E_8	F_4	G_2	$A_l(l \geq 1)$	$B_l(l \geq 2)$	$C_l(l \geq 3)$	$D_l(l \geq 4)$
$\dim(\mathfrak{g})$	78	133	248	52	14	l^2+2l	$2l^2+l$	$2l^2+l$	$2l^2-l$
$\mathcal{M}(\mathfrak{g})$	16	27	36	9	3	$\lfloor \frac{(l+1)^2}{4} \rfloor$	$\frac{l(l-1)}{2}+1$	$\frac{l(l+1)}{2}$	$\frac{l(l+1)}{2}$

Consequently, the problem of computing the maximal abelian dimension for semisimple Lie algebras is completely solved. In this sense, some authors have also studied this topic for solvable Lie algebras in order to solve the problem for all Lie algebras.

Ourselves have been widely studying the maximal abelian dimension of solvable Lie algebras for the last five years (see [9–11], for instance). The main goal is to know the main properties of this invariant and use them in order to solve several open problems, like the classification of Lie algebras in general.

In this way, Benjumea, Echarte, Núñez and Tenorio [3] began the study of abelian subalgebras of a given Lie algebra by using unipotent matrices to represent them in 2004. In that article, they dealt with abelian subalgebras of the nilpotent Lie algebra $\mathfrak{g}_n$, formed of $n \times n$ strictly upper triangular matrices. Let us note that these algebras $\mathfrak{g}_n$ are very interesting because every finite-dimensional nilpotent Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of some Lie algebra $\mathfrak{g}_n$.

They obtained an algorithmic method for computing a specific abelian Lie subalgebra of the Lie algebra g_n depending on the parity of the matrix order of the vectors constituting the basis of $\mathfrak{g}_n$. When n was even, a k^2-dimensional abelian subalgebra was obtained, with $n = 2k$; whereas when n was odd, a following (k^2+k)-dimensional abelian subalgebra of $\mathfrak{g}_n$ was computed, with $n = 2k + 1$.

To obtain these results, they introduced the concept of *maximal abelian dimension* of a fixed and given Lie algebra as the maximum among the dimensions of its abelian subalgebras and set the following conjecture.

Conjecture 5.1. *The maximal dimension of an abelian subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}_n$ *is given by*

$$\dim \mathfrak{h} = \begin{cases} k^2, & \text{if } n = 2k, \text{ with } k \in \mathbb{N}, \\ \dfrac{(2k+1)^2 - 1}{4}, & \text{if } n = 2k+1, \text{ with } k \in \mathbb{N}. \end{cases}$$

Moreover, they proved that: *An abelian subalgebra of* $\mathfrak{g}_n$ *cannot be obtained with dimension one unit less than the one corresponding with* $\mathfrak{g}_n$, which is an obstruction to obtain the corresponding representation of such abelian algebras as subalgebras of $\mathfrak{g}_n$. Consequently, they obtained the following corollary.

Corollary 5.1. *If* $n \in \mathbb{N}$ *with* $n \geq 4$, *then the simply connected Lie group associated with the abelian Lie algebra of dimension one unit less than the corresponding with* $\mathfrak{g}_n$, *cannot be represented as a Lie subgroup of* G_n *(where* G_n *is the Lie group naturally associated with* $\mathfrak{g}_n$*).*

Three years later, Benjumea, Núñez and Tenorio [4] completed the study begun in [3], obtaining the proof of Conjecture 5.1 settled in that previous article. To obtain this result, they distinguished between *main and non-main vectors* in a fixed basis of a given Lie algebra, using them to express the abelian subalgebras of the algebra. The authors also used the classical bound (given by Jacobson [32] and previously by Schur [58]) for the dimension of an abelian subalgebra in the Lie algebra $M_n(\mathbb{K})$, of $n \times n$ square matrices over a field $\mathbb{K}$, to give the following theorem.

Theorem 5.1. *Let* $\mathfrak{a}$ *be an abelian subalgebra of the Lie algebra* $M_n(\mathbb{K})$ *over an arbitrary field* $\mathbb{K}$. *Then* $\dim(\mathfrak{a}) \leq [\frac{n^2}{4}] + 1$, *where* $[x]$ *denotes the integer part of* x. *Moreover, there exists an abelian subalgebra whose dimension is exactly this bound.*

Moreover, they got two obstructions for the dimension of abelian subalgebras of the Lie algebra $\mathfrak{g}_n$: such subalgebras cannot be of dimension two or three unit less than the dimension of the Lie algebra $\mathfrak{g}_n$. Finally, they dealt with the general case in order to prove that there do not exist any abelian subalgebras of dimension greater than the dimension posed in Conjecture 5.1 and given in [3]. Indeed they obtained the following theorem.

Theorem 5.2. *Let us consider* $n \in \mathbb{N}$, *with* $n \geq 2$. *The maximal abelian dimension of the Lie algebra* $\mathfrak{g}_n$ *is given by*

$$\mathcal{M}(\mathfrak{g}_n) = \begin{cases} k^2, & \text{if } n = 2k, \text{ with } k \in \mathbb{N}, \\ k^2 + k, & \text{if } n = 2k+1, \text{ with } k \in \mathbb{N}, \end{cases}$$

which is an improvement in one unit for Schur–Jacobson's bound of the dimension of abelian subalgebras in the Lie algebra $\mathfrak{g}_n$.

Also in 2007, Núñez and Tenorio [51] dealt with the maximal abelian dimension of Heisenberg algebras. This type of Lie algebra constitutes one of the most structured and special subclasses of nilpotent Lie algebras. Studying them is very interesting because of its many several applications to both the theory of nilpotent Lie algebras and Mathematical Physics. The appearance of these algebras is set at the beginnings of the 20th century with the introduction of Quantum Mechanics. One of the most important physical applications of this family of Lie algebras is to determine the state of a particle in a given time t. Heisenberg took the position vector and its momentum vector, considering them as Hilbert operators. This structure led to a structure of Heisenberg algebra. Let us recall that the Heisenberg algebra $\mathfrak{H}_k$ is the $(2k+1)$-dimensional Lie algebra having a basis $\langle \{e_i\}_{i=1}^{2k+1} \rangle$, with the law

$$[e_{2i}, e_{2i+1}] = e_1, \quad \forall\, i = 1, \ldots, k.$$

In [51], the maximal abelian dimension of an arbitrary Heisenberg algebra was computed, using the technique of main and non-main vectors introduced in [4]. The maximal abelian dimension was computed by means of the proof of the following two theorems.

Theorem 5.3. *Given* $k \in \mathbb{N}$ *and* $\alpha \in \mathbb{N} \cup \{0\}$, *the maximal abelian dimension of* $\mathfrak{H}_k$ *is bounded by*

$$k + 1 \leq \mathcal{M}(\mathfrak{H}_k) \leq 2k - \alpha, \quad \forall\, k \geq \alpha + 1.$$

Theorem 5.4. *Given* $k \in \mathbb{N}$, *the maximal abelian dimension of* $\mathfrak{H}_k$ *is* $\mathcal{M}(\mathfrak{H}_k) = k + 1$. *Moreover, a* $(k+1)$-*abelian subalgebra in* $\mathfrak{H}_k$ *is* $\mathfrak{a}_k = \langle \{e_{2i+1}\}_{i=0}^{k} \rangle$.

Later, Ceballos, Núñez and Tenorio [8] determined in 2008 the maximal abelian dimension and hence the maximal abelian subalgebras of a particular family of solvable Lie algebras: the Lie algebras $\mathfrak{h}_n$, of $n \times n$ upper-triangular matrices. The importance of this type of Lie algebra lies in the fact that every finite-dimensional solvable Lie algebra $\mathfrak{s}$ is isomorphic to a subalgebra of some Lie algebra $\mathfrak{h}_n$. By using an algorithmic procedure constructed by them to compute abelian subalgebras of $\mathfrak{h}_n$ up to a certain dimension, they obtained that the dimension of such an abelian subalgebra is

$$B_n = \begin{cases} k^2 + 1, & \text{if } n = 2k, \\ k^2 + k + 1, & \text{if } n = 2\,k + 1. \end{cases}$$

Since B_n was equal to the upper bound of $\mathcal{M}(\mathfrak{h}_n)$ given by Jacobson [32] and Schur [58], the maximal abelian dimension of $\mathfrak{h}_n$ had to be B_n and the abelian subalgebras computed by the algorithm were maximal abelian subalgebras of $\mathfrak{h}_n$.

In 2008, the same authors [13] developed a new proof of Jacobson–Schur's bound for computing the maximal abelian dimension of the Lie algebras $\mathfrak{h}_n$, by using a theoretical method instead of the other techniques used in [8]. In this way, a new proof of the Jacobson–Schur's bound was given for the dimension of the largest abelian subalgebra of the Lie algebra $\mathfrak{h}_n$. The main result of the article was the following, whose proof was done by distinguishing five different cases depending on the number of non-main vectors coming from the last column in the matrix expression of the vectors in $\mathfrak{h}_n$.

Theorem 5.5. *Given $n \in \mathbb{N}$ with $n \geq 2$, the dimension of any abelian subalgebra of $\mathfrak{h}_n$ is $\dim(\mathfrak{h}_n) - r$, where $\dim(\mathfrak{h}_n) - B_n \leq r \leq \dim(\mathfrak{h}_n) - 1$.*

Other particular methods were applied by themselves [14] to compute the maximal abelian dimension of any given solvable Lie algebra of dimension less than 7. More concretely, they studied their maximal abelian dimension by applying an algorithmic method which allowed them to go ruling out non-valid values for the maximal abelian dimension until obtaining the exact value. Dimension 6 is the highest one classified for solvable Lie algebras by the mathematical community, because the classification of solvable Lie algebras is still an open problem (indeed, the authors used in their work the classical classification of solvable Lie algebras given by Mubarakzjanov [48–50] and Turkowsky [66]).

Finally, in 2009, the same authors [12] searched an algorithm which allowed to study and compute the maximal abelian subalgebra for any arbitrary Lie algebra (not necessarily solvable or nilpotent). This computational study was exemplified with the very special class of model filiform Lie algebras. The main reason to consider this family of Lie algebras laid in the fact that the law of this family has $n-2$ non-zero brackets for dimension n and its characteristic series is maximal.

At present, several authors continue dealing with this topic, obtaining several significative results. At this respect, in the following section we show a list of open problems whose research is being carried out.

6. Conclusions and Open Problems

Bearing in mind that the classification problem for solvable Lie algebras is still unsolved, we are going to comment some general open problems interesting (in our opinion) to advance some steps in the classification of these algebras.

The main problem we would like to mention is the following: *let $\mathfrak{g}$ be an n-dimensional Lie algebra which verifies that the largest dimension of an abelian subalgebra is k. We ask ourselves for some conditions or properties necessaries to affirm if $\mathfrak{g}$ is solvable or semisimple or non-solvable or non-semisimple*, *for example*; *i.e. studying which is the structure underlying to the Lie algebra $\mathfrak{g}$*.

In this way, Tenorio [62] had previously studied the following problem: *Fixed and given a Lie algebra of dimension n and maximal abelian dimension $n-1$, what type of Lie algebra does it belong to*? To answer this question, the author used solvable, nilpotent and filiform Lie algebras. He gave several necessary and sufficient conditions for this problem. Let us note that, in the other previous papers, the problem analyzed was to compute the maximal abelian dimension of a fixed Lie algebra. In this case, Tenorio dealt with the converse problem. As a sake of example of the criteria given in this article, we recall the following two sufficient and necessary conditions.

Theorem 6.1. *Let $\mathfrak{g}$ be an n-dimensional complex nilpotent Lie algebra satisfying $\mathcal{M}(\mathfrak{g}) = n-1$. Then there exists an ordered sequence $(s_1,\ldots,s_p)$ such that $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}_{s_1,\ldots,s_p}$ defined by the following law:*

$$\begin{cases} [Y,X_i^1] = X_{i+1}^1, \ \textit{with } i=1,\ldots,s_1-1, \ [Y,X_{s_1}^1]=0 \\ [Y,X_i^2] = X_{i+1}^2, \ \textit{with } i=1,\ldots,s_2-1, \ [Y,X_{s_2}^2]=0 \\ \vdots \\ [Y,X_i^p] = X_{i+1}^p, \ \textit{with } i=1,\ldots,s_p-1, \ [Y,X_{s_p}^p]=0. \end{cases}$$

Corollary 6.1. *Let $\mathfrak{g}$ be an n-dimensional complex filiform Lie algebra. Its maximal abelian dimension is $n-1$ if and only if $\mathfrak{g}$ is isomorphic to the model filiform Lie algebra.*

Recently, the authors of this survey are studying the generalization of those results in order to solve the converse problem when the maximal abelian dimension of the n-dimensional Lie algebra $\mathfrak{g}$ is $n-k$ for $k \geq 2$.

At this respect, the starting point of their research on this converse problem is based on the following result which is already proved.

Lemma 6.1. *Let $\mathfrak{g}$ be an n-dimensional complex Lie algebra and $\mathcal{M}(\mathfrak{g}) = n-k$, with $n > k > 1$. If $\{e_h\}_{h=1}^n$ is a basis of $\mathfrak{g}$ verifying the following conditions:*

(1) *$\langle e_{k+1},\ldots,e_n\rangle$ is a maximal abelian subalgebra;*
(2) *the structure constants $c_{i,j}^h$ and $c_{k,k+1}$ are known, where $j > i$ and $1 \leq i \leq k+1$;*
(3) *the structure constant $c_{1,k+1}^k$ is non-zero,*

then, the law of the Lie algebra $\mathfrak{g}$ is completely determined.

Another open problem is to compare and contrast the properties of the maximal dimension for abelian subalgebras and abelian ideals. To consider abelian ideals, more restrictive hypotheses are needed and it is sometimes possible to obtain better results with this consideration than dealing with abelian subalgebras. This is due to the fact that if we consider an abelian ideal, we can define its quotient with the Lie algebra as well as its representation with the adjoint map restricted to the abelian ideal. Therefore, we think it would be also interesting to study the converse problem when abelian ideals are considered instead of abelian subalgebras and try to find conditions such that both considerations are equivalent.

To finish the present article, we would finally like to point out that some authors (like Laffey [42], for instance) are also dealing with the problem of the minimal dimension of maximal abelian subalgebras, although at present no relation between both studies has been found.

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