A distributed set-membership estimator for linear systems with reduced computational requirements

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Abstract

In this paper, a distributed set-membership estimator for linear full-coupled systems affected by bounded disturbances is presented. The estimator uses a multi-hop staircase decomposition, capturing the locally unobservable subspaces in a cascaded fashion with the information incoming from other agents involved in the network. Each agent has to find different sets for each subspace, that are mathematically described by zonotopes. The observer gains that minimize the size of those sets, i.e. the estimation uncertainty, can be designed in independent distributed steps by means of simple algebraic equations. Simulations are given to compare the proposed solution with others in the field. An important benefit of the proposed structure is the reduction of the computational requirements with respect to existing solutions.

Key words: Distributed set-membership estimation, Multi-agent systems, Linear system, Zonotopes, Multi-hop decomposition.

1 Introduction

The application of a significant amount of techniques, in particular, advanced control or fault-detection, are based on a state-space description of the system. In most cases, the information about the current state of the system is not available, therefore it is necessary to resort to estimators of the plant states.

Moreover, the growing number of embedded sensors dispersed in complex systems is promoting the application of distributed estimation schemes. In Ge, Han, Zhang, Ding & Yang (2019), Ierardi et al. (2019), Zou et al. (2020) different distributed estimators are surveyed with a variety of focuses. Distributed event-triggered mechanisms for estimation are analysed in Ge, Han, Zhang, Ding & Yang (2019); different techniques applied to cyber-physical systems are studied in Ierardi et al. (2019); and the recent advances of moving horizon estimation are summarized in Zou et al. (2020).

A wide variety of observers and estimation techniques have been presented in the literature. There are many works based on distributed versions of the Kalman Filter, such as Chen et al. (2019), He et al. (2020). Other approaches adapt the Luenberger observer to distributed paradigms using consensus techniques del Nozal et al. (2019), Mitra & Sundaram (2018). The application of H_∞ theory Ge, Han & Wang (2019), Ugrinovskii & Fridman (2014), moving horizon schemes Chen et al. (2017), Farina et al. (2010), Yin & Liu (2017) or bayesian approaches Battistelli & Chisci (2014), have also obtain remarkable results.

Notwithstanding, the literature concerning distributed set-membership estimation is more scarce. Most of the references make use of ellipsoids, intervals or zonotopes to describe the sets. In general, zonotopic formulations help to reduce the computational load compared to ellipsoidal formulations, because many operations, such as linear transformations and summations, can be easily computed with simple algebraic manipulations. On the other hand, zonotopes usually obtain more precise estimations than those estimators based on intervals, Alanwar et al. (2020), Tang et al. (2019).

The review of the state of the art begins with distributed estimators with ellipsoidal descriptions. The authors in Liu et al. (2020) use the Round-Robin communication protocol and the multi-rate strategy to regulate the different sampling rates in the estimation performance. An event-based communication mechanism over sensor networks is presented in Ma et al. (2016). The work Xia et al. (2018) proposes a distributed set-membership es-

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timation algorithm in which the ellipsoids locally computed by all the agents are merged in a global central unit. Finally, interval observers have found application in the estimation of distributed power systems Zhang et al. (2020), and for spatially distributed systems described by partial differential equations Kharkovskaia et al. (2020).

Regarding distributed zonotopic estimation most works deal with the estimation of a system described by a set of interconnected subsystems, and the estimation goal of each agent is to construct sets containing the states of the local subsystem Combastel & Zolghadri (2018), Orihuela, Millán, Roshany-Yamchi & García (2018), Orihuela, Roshany-Yamchi, García & Millán (2018), Wang et al. (2017), Wang et al. (2018). Instead, the papers Alanwar et al. (2020), García et al. (2020) present estimators able to reconstruct and wrap the whole state vector. The former proposes the use of diffusion techniques and in García et al. (2020) they tackle the joint problem of distributed estimation and control.

These two last papers Alanwar et al. (2020), García et al. (2020), together with Xia et al. (2018), constitute the unique references, from the best of our knowledge, that cope with the same problem as the one studied here, this is, to present a guaranteed distributed estimation method for full-coupled linear perturbed systems to be implemented in a set of distributed agents that need to communicate and collaborate to achieve this goal. A novel algorithm is proposed for this problem aiming at reducing the computational requirements. The main contributions of this paper are summarized next:

- The multi-hop staircase decomposition in del Nozal et al. (2019) is used, so the dynamics of the system can be rewritten allowing to present a cascaded implementation of the distributed set-membership observer.
- According to the previous transformation, each agent is required to compute now a family of sets, one for each subspace, instead of a unique but larger set for the complete state space.
- The observer structure aims to minimize the estimation uncertainty computing adequate local and neighbour gains. These gains can be designed in independent and simple distributed steps, through the solution of algebraic equations.

The proposed method is able to reduce the computational requirements compared to existing solutions, keeping a similar estimation performance. This work extends our preliminary result in the IFAC WC Orihuela et al. (2020), in which only two agents were considered.

This paper is organized as follows. Section 2 presents some notation and preliminaries on zonotopes and on the multi-hop staircase decomposition. The system definition and the problem statement are presented in Section 3. Section 4 describes the proposed distributed set-membership observer. Section 5 presents some considerations about computational requirements. Simulations and numerical results are illustrated in Section 6. Conclusions and future works are drawn in Section 7.

Notation. Given a set of matrices A_i, for $i = 1, \dots, n$, of appropriate dimensions, operator $\operatorname*{cat}_{i=1}^{n}\{A_i\}$ implies the concatenation of the matrices, that is, $\operatorname*{cat}_{i=1}^{n}\{A_i\} = [A_1\ A_2\ \cdots\ A_n]$. The operator blkdiag$(A_i)$ returns the block diagonal matrix created by aligning the input matrices $(A_1, A_2, ..., A_n)$ along the diagonal.

2 Preliminaries on zonotopes

Definition 1. A zonotope $\mathcal{X}$, denoted with calligraphic, capital letters, is a centrally symmetric, convex set determined by its center $c \in \mathbb{R}^n$, and by a matrix $H \in \mathbb{R}^{n\times q}$: $\mathcal{X} = \langle c, H\rangle = \{c + \sum_{i=1}^{q} \varsigma_i h_i : \forall i,\ |\varsigma_i| \le 1\}$, where $h_i \in \mathbb{R}^n$ (columns of H) are called *generator vectors*.

The *order* of a zonotope is given by the number of generator vectors. The *F-radius* of a zonotope is the Frobenius norm of its generator matrix H, this is, $||H||_F = \sqrt{tr(H^T H)}$. The *covariation* of a zonotope is defined as $P_{\mathcal{X}} = HH^T$ (see Combastel (2015)). Let $\mathcal{X} = \langle c, H\rangle$ and $\mathcal{Y}_i = \langle c_i, H_i\rangle$, $i = 1, \dots, n$, be respectively a zonotope and a set of zonotopes, and let R be a matrix of appropriate dimensions. A linear transformation of a zonotope is given by $R\mathcal{X} = \langle Rc, RH\rangle$; the Minkowski sum of several zonotopes is obtained as $\bigoplus_{i=1}^{n} \mathcal{Y}_i = \bigoplus_{i=1}^{n} \langle c_i, H_i\rangle = \langle c_y, H_y\rangle$, with $c_y = \sum_{i=1}^{n} c_i$ and $H_y = \text{cat}_{i=1}^{n}\{H_i\}$. Given a matrix A and any vectors such that $x \in \mathcal{X}$ and $w \in \mathcal{W}$, it holds that $y := Ax + w \in A\mathcal{X} \oplus \mathcal{W}$. The operator $red_q(\mathcal{X})$, is an order reduction of the zonotope $\mathcal{X}$ in such a way that $\mathcal{X} \subseteq red_q(\mathcal{X})$, and the order of $red_q(\mathcal{X})$ is q. There are many methods to reduce the order of the zonotope (see Kopetzki et al. (2017), Combastel (2015)). This paper uses the one in Combastel (2015).

3 Problem statement

In this paper, we consider the problem of set-membership state estimation for a multi-output linear system affected by bounded disturbances, observed by p agents connected according to a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The system is described by the following equations:

$$x(k+1) = Ax(k) + Dw(k), \tag{1}$$
$$y_i(k) = C_i x(k) + v_i(k),\ i = 1, \dots, p, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $y_i \in \mathbb{R}^{m_i}$ is the output locally measured by each agent, $w \in \mathbb{R}^r$ represents disturbances or unmodeled dynamics, and $v_i \in \mathbb{R}^v$ are measurement noises. Matrices A, D, C_i are constant matrices of appropriate dimensions.

Assumption 1. Disturbances and noises belong to known bounded sets described as zero-centered zonotopes, this is, $w(k) \in \mathcal{W} = \langle 0, Q\rangle$, $v_i(k) \in \mathcal{V}_i = \langle 0, R_i\rangle$, being Q and R_i known matrices of adequate dimensions.

Assumption 2. System (1)-(2) is collectively detectable (see Definition 5 in del Nozal et al. (2019)).

Assumption 2 states that, for all agents, there exists enough information and links in the graph so that they can all reconstruct the whole state vector with local measurements and neighboring incoming information.

According to the multi-hop decomposition described in del Nozal et al. (2019) and Assumption 2, for each agent $i = 1, \dots, p$, it is possible to find a coordinate transformation matrix $T_i \in \mathcal{R}^{n\times n}$ such that under the change of variables $z_i = T_i^\top x$, system (1)-(2) can be transformed into the observability staircase form:

$$z_i(k) = \begin{bmatrix} W_{i,\ell_i+1}^\top \\ W_{i,\ell_i}^\top \\ \vdots \\ W_{i,\rho+1}^\top \\ W_{i,\rho}^\top \\ \vdots \\ W_{i,0}^\top \end{bmatrix} x(k) = \begin{bmatrix} z_{i,\ell_i+1}(k) \\ z_{i,\ell_i}(k) \\ \vdots \\ z_{i,\rho+1}(k) \\ z_{i,\rho}(k) \\ \vdots \\ z_{i,0}(k) \end{bmatrix}, \quad (3)$$

where $W_{i,\rho} \in \mathbb{R}^{n\times n_{i,\rho}}$ is the innovation matrix of agent i at hop ρ, and $z_{i,\rho}(k) \in \mathbb{R}^{n_{i,\rho}}$ is the transformed state for agent i at hop ρ. Matrix W_{i,ℓ_i+1} defines the basis of the collectively unobservable but detectable subspace. The transformed state vector $z_i(k)$ comprises the locally observable modes $z_{i,0}(k)$ and the locally unobservable modes, that are locally observable by the other agents in the corresponding ρ-hop reachable set of agent i. Using $T_i^{-1} = T_i^\top$, it holds:

$$x(k) = T_i z_i(k) = \sum_{r=0}^{\ell_i+1} W_{i,r} z_{i,r}(k). \quad (4)$$

Using equation (1), we obtain:

$$z_i(k+1) = T_i^\top A T_i z_i(k) + T_i^\top D w(k), \quad (5)$$

with $T_i^\top A T_i$ upper block diagonal according to Proposition 9 in del Nozal et al. (2019). Then, the dynamics at each hop is given by:

$$z_{i,\rho}(k+1) = \sum_{r=0}^{\rho} W_{i,\rho}^\top A W_{i,r} z_{i,r}(k) + W_{i,\rho}^\top D w(k). \quad (6)$$

Agents have the mission of estimating the state of the system (1), having limited information about it (2), and communication capabilities with others also limited to their local neighbourhood. With the information available at any time instant k, let's define the *a priori* and *a posteriori* zonotopes for each agent and hop, intended to contain the actual state of each subspace at instant $k+1$ and k, respectively:

- *A priori*: $\hat{\mathcal{Z}}_{i,\rho}(k+1|k) = \langle c_{i,\rho}(k+1|k), H_{i,\rho}(k+1|k)\rangle$,
- *A posteriori*: $\hat{\mathcal{Z}}_{i,\rho}(k|k) = \langle c_{i,\rho}(k|k), H_{i,\rho}(k|k)\rangle$.

Problem 1. Consider a system described by (1), a set of agents measuring the outputs (2) and connected according to a given communication graph $\mathcal{G}$. Under Assumptions 1 and 2, and relying only on locally available information, each agent must:

a) Find sets, by means of zonotopes, in which the actual state of its associated subspaces are continuously contained. In particular, *a posteriori* and *a priori* sets must be found such that $z_{i,\rho}(k) \in \hat{\mathcal{Z}}_{i,\rho}(k|k)$ and $z_{i,\rho}(k+1) \in \hat{\mathcal{Z}}_{i,\rho}(k+1|k)$, $\forall i, \rho, k$.

b) Minimize the estimation uncertainty, measured through the *F-radius* of the *a posteriori* zonotopes, as defined in Section 2.

The main benefit of the proposed structure is the reduction of the computational requirements of the system, as it uses the minimum information necessary to reconstruct the whole state. This is achieved by discarding, at every hop ρ, redundant information of those subspaces that were already estimated at previous hops, thus avoiding the inclusion of unnecessary data in the estimation process.

4 Distributed set-membership estimator

In this section we will present the estimator for the locally-observable subspace, which is based on the preliminary work Orihuela et al. (2020). Later, we present the estimator for the locally-unobservable subspaces, which constitutes the main contribution of the paper.

4.1 Estimation of the locally-observable subspaces

The *a posteriori* estimation set for hop $\rho = 0$, this is $\hat{\mathcal{Z}}_{i,0}(k|k) = \langle c_{i,0}(k|k), H_{i,0}(k|k)\rangle$ is defined as:

$$\begin{aligned} c_{i,0}(k|k) &= c_{i,0}(k|k-1) \\ &\quad + L_i(k)\left(y_i(k) - C_i W_{i,0} c_{i,0}(k|k-1)\right), \quad (7) \\ H_{i,0}(k|k) &= [\left(I - L_i(k) C_i W_{i,0}\right) H_{i,0}(k|k-1), \quad (8) \\ &\quad -L_i(k) R_i], \end{aligned}$$

where $L_i(k) \in \mathbb{R}$ is the local gain to be designed. On the other hand, the *a priori* estimation set for hop $\rho = 0$, this is $\hat{\mathcal{Z}}_{i,0}(k+1|k) = \langle c_{i,0}(k+1|k), H_{i,0}(k+1|k)\rangle$, is described as:

$$c_{i,0}(k+1|k) = W_{i,0}^\top A W_{i,0} c_{i,0}(k|k), \tag{9}$$

$$H_{i,0}(k+1|k) = [W_{i,0}^\top A W_{i,0} H_{i,0}(k|k), \quad W_{i,0}^\top DQ]. \tag{10}$$

Theorem 1 *Orihuela et al. (2020) Let's assume that $z_{i,0}(k_o) \in \hat{\mathcal{Z}}_{i,0}(k_o|k_o-1), \forall i$, for some particular instant k_o. Then, if the distributed set-membership observer in* (7)-(10) *is implemented by every agent, Problem 1a is solved for hop $\rho = 0$ and for any possible realization of $L_i(k)$, for all $k > k_o$.*

The information locally available for agent i includes the *a priori* zonotope $\hat{\mathcal{Z}}_{i,0}(k|k-1)$, the output $y_i(k)$, and the set $\mathcal{V}_i = \langle 0, R_i\rangle$. Therefore, any agent is able to distributedly compute the *a posteriori* zonotope at any instant. The new *a priori* set $\hat{\mathcal{Z}}_{i,0}(k+1|k)$ depends on the *a posteriori* zonotope and the set $\mathcal{W} = \langle 0, Q\rangle$, which are also locally available. Now, for the solution of Problem 1b, the observer gains $L_i(k)$ must be properly designed.

Theorem 2 *Orihuela et al. (2020) Consider that each agent implements the observer structure* (7)-(10)*. Then, the local observer gain $L_i(k)$ that solve Problem 1b for hop $\rho = 0$ is given by:*

$$L_i(k) = P_{\hat{\mathcal{Z}}_{i,0}(k|k-1)} W_{i,0}^\top C_i^\top \times \Big(C_i W_{i,0} P_{\hat{\mathcal{Z}}_{i,0}(k|k-1)} W_{i,0}^\top C_i^\top + P_{\mathcal{V}_i}\Big)^{-1}, \tag{11}$$

where $P_{\mathcal{Z}}$ stands for the covariation of zonotope $\mathcal{Z}$ as defined in Definition 1.

4.2 Estimation of the locally-unobservable subspaces

In order to construct estimation sets for the locally-unobservable subspaces, each agent must rely on the information provided by neighbouring agents, combining it with the *a priori* set locally known.

Assumption 3. The agents are allowed to communicate once during each sampling period to their neighbourhood according to a given graph $\mathcal{G}$.

Attending to Assumption 3, each agent will transmit at once (at the same time) the available estimation sets of all its hops after the estimation of the locally-observable subspace have taken place. This is, each agent will send to its neighboring agents the *a posteriori* estimation set for hop $\rho = 0$, i.e. $\hat{\mathcal{Z}}_{i,0}(k|k)$, and the *a priori* estimation sets for hops $\rho = 1, \ldots, \ell_i$, i.e. $\hat{\mathcal{Z}}_{i,\rho}(k|k-1)$.

Having received similar information from their neighbours, agent i computes the *a posteriori* estimation set for hops $\rho = 1, \ldots, \ell_i + 1$, $\hat{\mathcal{Z}}_{i,\rho}(k|k)$, as:

$$\begin{aligned} c_{i,\rho}(k|k) = &\Bigg(I - \sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top W_{i,\rho}\Bigg) c_{i,\rho}(k|k-1) \\ &- \sum_{r=0}^{\rho-1} \Bigg(\sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top\Bigg) W_{i,r} c_{i,r}(k|k) + \\ &\sum_{j\in N_i} N_{i,j,\rho}(k) c_{j,\rho-1}(k|k-1), \end{aligned} \tag{12}$$

$$\begin{aligned} H_{i,\rho}(k|k) = &\Bigg[\Bigg(I - \sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top W_{i,\rho}\Bigg) H_{i,\rho}(k|k-1), \\ &- \operatorname*{cat}_{r=0}^{\rho-1} \Bigg\{\Bigg(\sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top\Bigg) W_{i,r} H_{i,r}(k|k)\Bigg\}, \\ &\operatorname*{cat}_{j\in N_i} \{N_{i,j,\rho}(k) H_{j,\rho-1}(k|k-1)\}\Bigg], \end{aligned} \tag{13}$$

where $N_{i,j,\rho}(k) \in \mathbb{R}^{n_{i,\rho}\times n_{j,\rho-1}}$ are the neighbour observer gains to be designed.

The *a priori* estimation set for hops $\rho = 1, \ldots, \ell_i + 1$ will be found as:

$$c_{i,\rho}(k+1|k) = \sum_{r=0}^{\rho} W_{i,\rho}^\top A W_{i,r} c_{i,r}(k|k), \tag{14}$$

$$H_{i,\rho}(k+1|k) = \Bigg[\operatorname*{cat}_{r=0}^{\rho} \{W_{i,\rho}^\top A W_{i,r} H_{i,r}(k|k)\}, W_{i,\rho}^\top DQ\Bigg]. \tag{15}$$

Theorem 3 *Let's assume that $z_{i,0}(k_o) \in \hat{\mathcal{Z}}_{i,0}(k_o|k_o)$, and $z_{i,\rho}(k_o) \in \hat{\mathcal{Z}}_{i,\rho}(k_o|k_o-1), \forall i, \rho > 0$ for some particular instant k_o. Then, if the distributed set-membership observer in* (12)-(15) *is implemented by every agent, Problem 1a is solved for hops $\rho = 1, \ldots, \ell_i + 1$, and for any possible realization of $N_{i,j,\rho}(k)$, for all $k > k_o$.*

PROOF. The theorem will be proven by induction. Consider $z_{i,0}(k) \in \hat{\mathcal{Z}}_{i,0}(k|k)$ for all i, and $z_{i,\rho}(k) \in \hat{\mathcal{Z}}_{i,\rho}(k|k-1)$ for all i and for hops $\rho = 1, \ldots, \ell_i + 1$ at some instant k. It is satisfied:

$$\begin{aligned} z_{i,\rho}(k) = z_{i,\rho}(k) + \sum_{j\in N_i} N_{i,j,\rho}(k) \Big(&W_{j,\rho-1}^\top \hat{x}_j(k) \\ &- W_{j,\rho-1}^\top x(k) + W_{j,\rho-1}^\top e_j(k)\Big), \end{aligned}$$

being $e_j(k) \triangleq x(k) - \hat{x}_j(k)$. Note that the term $W_{j,\rho-1}^\top \hat{x}_j(k)$ plays here the role of $y_i(k)$ in hop $\rho = 0$. This is the information that agent i receives and uses to compute the estimation set. Whereas the measured output $y_i(k)$ was contaminated with noise $v_i \in \mathcal{V}_i$, this signal $W_{j,\rho-1}^\top \hat{x}_j(k)$, representing the states that the neighbour observes at its previous hop, is affected by the uncertainty of the corresponding zonotope. Now, using (4),

$$z_{i,\rho}(k) = z_{i,\rho}(k) + \sum_{j\in N_i} N_{i,j,\rho}(k)\Big(W_{j,\rho-1}^\top \hat{x}_j(k) - W_{j,\rho-1}^\top \sum_{r=0}^{\ell_i} W_{i,r} z_{i,r}(k) + W_{j,\rho-1}^\top e_j(k)\Big),$$

and after some mathematical manipulations and according to Lemma 3 in del Nozal et al. (2019), this is $W_{j,\rho-1}^\top W_{i,r} = 0, \forall r > \rho$, it can be obtained:

$$z_{i,\rho}(k) = \Big(I - \sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top W_{i,\rho}\Big) z_{i,\rho}(k) - \sum_{r=0}^{\rho-1}\left(\sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top\right) W_{i,r} z_{i,r}(k) + \sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top \Big(\hat{x}_j(k) + e_j(k)\Big).$$

For $\rho = 1$, it has been assumed $z_{i,1}(k) \in \hat{\mathcal{Z}}_{i,1}(k|k-1)$ and $z_{i,0}(k) \in \hat{\mathcal{Z}}_{i,0}(k|k)$, and finally agreeing that $\hat{x}_j + e_j = x$ from which it derives $W_{j,0}^\top(\hat{x}_j(k) + e_j(k)) \in \hat{\mathcal{Z}}_{j,0}(k|k-1)$ using (3), the states are contained in:

$$z_{i,1}(k) \in \left(I - \sum_{j\in N_i} N_{i,j,1}(k) W_{j,0}^\top W_{i,1}\right) \hat{\mathcal{Z}}_{i,1}(k|k-1) \oplus \left(- \sum_{j\in N_i} N_{i,j,1}(k) W_{j,0}^\top W_{i,0} \hat{\mathcal{Z}}_{i,0}(k|k)\right) \oplus \left(\bigoplus_{j\in N_i} N_{i,j,1}(k) \hat{\mathcal{Z}}_{j,0}(k|k-1)\right).$$

Previous equation ensures that $z_{i,1}(k) \in \hat{\mathcal{Z}}_{i,1}(k|k)$, defined in equations (12)-(13).

Repeating the same ideas for subsequent hops, since $z_{i,\rho}(k) \in \hat{\mathcal{Z}}_{i,\rho}(k|k-1)$ holds for assumption, and $z_{i,r}(k) \in \hat{\mathcal{Z}}_{i,r}(k|k), \forall r = [0, \rho-1]$, has been proved for previous hops with the cascade structure, and knowing that $W_{j,\rho-1}^\top(\hat{x}_j(k) + e_j(k)) \in \hat{\mathcal{Z}}_{j,\rho-1}(k|k-1)$ using (3), the states are contained in:

$$z_{i,\rho}(k) \in \left(I - \sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top W_{i,\rho}\right) \hat{\mathcal{Z}}_{i,\rho}(k|k-1) \oplus \left(\bigoplus_{r=0}^{\rho-1} \left(- \sum_{j\in N_i} N_{i,j,\rho}(k) W_{j,\rho-1}^\top\right) W_{i,r} \hat{\mathcal{Z}}_{i,r}(k|k)\right) \oplus \left(\bigoplus_{j\in N_i} N_{i,j,\rho}(k) \hat{\mathcal{Z}}_{j,\rho-1}(k|k-1)\right).$$

Therefore, $z_{i,\rho}(k)$ has been proven to be contained in a zonotope described by equations (12)-(13).

Regarding the *a priori* sets, taking into account the evolution of the system in (6),

$$z_{i,\rho}(k+1) = W_{i,\rho}^\top A \sum_{r=0}^{\rho} W_{i,r} z_{i,r}(k) + W_{i,\rho}^\top D w(k),$$

and using $z_{i,r}(k) \in \hat{\mathcal{Z}}_{i,r}(k|k), \forall r = [0, \rho]$ and $w(k) \in \mathcal{W}$ by Assumption 1, it is possible to find a set containing $z_{i,\rho}(k+1)$:

$$z_{i,\rho}(k+1) \in W_{i,\rho}^\top A \bigoplus_{r=0}^{\rho} \Big(W_{i,r} \hat{\mathcal{Z}}_{i,r}(k|k)\Big) \oplus W_{i,\rho}^\top D \mathcal{W},$$

whose center and generator matrix are the ones in (14)-(15). Since $z_{i,0}(k) \in \hat{\mathcal{Z}}_{i,0}(k|k)$, $z_{i,\rho}(k_0) \in \hat{\mathcal{Z}}_{i,\rho}(k_0|k_0-1), \forall i, \rho$, holds for some particular instant k_0, then Problem 1a will be solved for $k > k_0$ by induction. □

Using the results of Theorem 3 and according to Assumptions 1 and 3, the *a priori* and *a posteriori* zonotopes can also be computed locally with the information available for each agent i, which includes the set $\mathcal{W}$, the zonotopes $\hat{\mathcal{Z}}_{i,\rho-1}(k|k)$ and $\hat{\mathcal{Z}}_{i,\rho}(k|k-1)$, and the received zonotopes $\hat{\mathcal{Z}}_{j,\rho-1}(k|k)$, for all neighbours. Please note that these sets are computed in an iterative way, from $\rho = 1$ to $\rho = \ell_i + 1$, because of the cascaded structure appearing in equations (12)-(15).

Remark. Assuming that more than one transmission might take place during each sampling period, relaxing Assumption 3 as in a multi-rate communication scheme Orihuela, Roshany-Yamchi, García & Millán (2018), the estimation algorithm can be implemented in a cascaded fashion, so that each agent builds and transmits the estimated set at each hop. Then, the *a posteriori* estimation sets $\hat{\mathcal{Z}}_{j,\rho-1}(k|k)$ could be used in (12)- (13), instead of the *a priori* sets $\hat{\mathcal{Z}}_{j,\rho-1}(k|k-1)$, which would improve the performance.

For the solution of Problem 1b, the observer gains $N_{i,j,\rho}(k)$ must be properly designed.

Theorem 4 *Consider that each agent implements the observer structure* (12)-(15). *The neighbour gains* $N_{i,j,\rho}(k)$ *that solve Problem 1b for hops* $\rho = 1, \dots, \ell_i + 1$, *are:*

$$N_{i,\rho}(k) \triangleq \underset{j \in N_i}{\text{cat}} \{N_{i,j,\rho}(k)\} = \\ = P_{\hat{\mathcal{Z}}_{i,\rho}(k|k-1)} W_{i,\rho}^\top \bar{W} \left(\bar{W}^\top \bar{P} \bar{W} + \bar{G} \right)^{-1}, \quad (16)$$

with:

$$\bar{G} = \underset{j \in N_i}{\text{blkdiag}} \, P_{\hat{\mathcal{Z}}_{j,\rho-1}(k|k-1)}, \quad (17)$$

$$\bar{P} = W_{i,\rho} P_{\hat{\mathcal{Z}}_{i,\rho}(k|k-1)} W_{i,\rho}^\top + \sum_{r=0}^{\rho-1} W_{i,r} P_{\hat{\mathcal{Z}}_{i,r}(k|k)} W_{i,r}^\top, \quad (18)$$

$$\bar{W} = \underset{j \in N_i}{\text{cat}} \{W_{j,\rho-1}\}, \quad (19)$$

where $P_{\mathcal{X}}$ *denote the covariation of zonotope* $\mathcal{X}$, *defined in Section 2.*

PROOF. The *F-radius* of $\hat{\mathcal{Z}}_{i,\rho}(k|k)$ is the Frobenius norm of $H_{i,\rho}(k|k)$. The argument $N_{i,\rho}(k)$ that minimize the *F-radius* of $\hat{\mathcal{Z}}_{i,\rho}(k|k)$ is the same argument that minimize $J_{i,\rho}(k) \triangleq tr(P_{\hat{\mathcal{Z}}_{i,\rho}(k|k)})$, for hops $\rho = 1, \dots, \ell_i + 1$. Function $J_{i,\rho}(k)$ is convex with respect to $N_{i,\rho}(k)$, so the gain that minimize $J_{i,\rho}(k)$ hold:

$$N_{i,\rho}^*(k) = \arg \left(\frac{\partial J_{i,\rho}(k)}{\partial N_{i,\rho}(k)} = 0 \right). \quad (20)$$

From equation (13), the covariation of the corresponding zonotope at hops $\rho = 1, \dots, \ell_i + 1$ is given by:

$$P_{\hat{\mathcal{Z}}_{i,\rho}(k|k)} = P_{\hat{\mathcal{Z}}_{i,\rho}(k|k-1)} - N_{i,\rho}(k) \bar{W}^\top W_{i,\rho} P_{\hat{\mathcal{Z}}_{i,\rho}(k|k-1)} \\ - P_{\hat{\mathcal{Z}}_{i,\rho}(k|k-1)} W_{i,\rho}^\top \bar{W} N_{i,\rho}^\top(k) + N_{i,\rho}(k) \bar{G} N_{i,\rho}^\top(k) \\ + N_{i,\rho}(k) \bar{W}^\top \bar{P} \bar{W} N_{i,\rho}^\top(k),$$

where the definitions in (17)-(19) have been used. Now, from equation (20), taking derivatives with respect to the trace, it yields:

$$\frac{\partial tr\left(P_{\hat{\mathcal{Z}}_{i,\rho}(k|k)}\right)}{\partial N_{i,\rho}(k)} = 2\bar{W}^\top \bar{S} \bar{W} N_{i,\rho}(k)^\top \\ - 2\bar{W}^\top W_{i,\rho} P_{\hat{\mathcal{Z}}_{i,\rho}(k|k-1)} + 2\bar{G} N_{i,\rho}(k)^\top \\ + 2\bar{W}^\top \bar{P} \bar{W} N_{i,\rho}(k)^\top. \quad (21)$$

By solving (21) the optimal neighbour observer gain in (16) is obtained to minimize the *F-radius* of the *a posteriori* zonotope $\hat{\mathcal{Z}}_{i,\rho}(k|k)$.
In a way similar to Theorem 3, the neighbour observer gains must be obtained in a cascaded way, from $\rho = 1$ to $\rho = \ell_i + 1$, so that term (18) can be computed. Then, Problem 1b for all hops is solved. □

From Theorem 4, and after computing the gains (16), each agent has to pick the corresponding columns of $N_{i,\rho}(k)$ to build $N_{i,j,\rho}(k)$ for each neighbour and hop. For the sake of clarity and ease of implementation, the iterative procedure to be implemented in each agent is presented in Algorithm 1.

Algorithm 1: Estimation loop

0. Initial zonotopes $\hat{\mathcal{Z}}_{i,0}(k|k-1)$, $\hat{\mathcal{Z}}_{i,\rho}(k|k-1), \forall \rho > 0$
1. Measurement $y_i(k)$ with (2)
2. *A posteriori* estimation
  2.1 Local gain $L_i(k)$ with (11)
  2.2 Zonotope $\hat{\mathcal{Z}}_{i,0}(k|k)$ with (7)-(8)
3. Communication
  3.1 Send $\hat{\mathcal{Z}}_{i,0}(k|k)$ $\hat{\mathcal{Z}}_{i,\rho}(k|k-1), \forall \rho > 0$
4. *A posteriori* estimation
  4.1 Neighbour gain $N_{i,j,\rho}(k)$ with (16)
  4.2 Zonotope $\hat{\mathcal{Z}}_{i,\rho}(k|k), \forall \rho > 0$ with (12)-(13)
5. *A priori* estimation
  5.1 Zonotope $\hat{\mathcal{Z}}_{i,0}(k+1|k)$ with (9)-(10),
    $\hat{\mathcal{Z}}_{i,\rho}(k+1|k), \forall \rho > 0$ with (14)-(15)
  5.2 Order reduction $red_q(\hat{\mathcal{Z}}_{i,0}(k+1|k))$,
    $red_q(\hat{\mathcal{Z}}_{i,\rho}(k+1|k)), \forall \rho > 0$

5 Computational requirements

This section compares the computational requirements of several set-membership estimation algorithms proposed. It only analyses the computational requirements when leveraging the information of all neighbours, as the estimation of the locally-observable subspaces has similar complexity in all cases.

All the methods present an optimization problem trying to reduce the volume or size of the estimation sets, merging the information received from the neighbours. In all the cases, the complexity of the method is P, this is, they can all be solved in polynomial time. However, the number of variables and the kind of optimization problem vary in each work.

The computational requirements are listed in Table 1. It is denoted by $\mathcal{O}(n_1, n_2)$ when a matrix of dimension $n_1 \times n_2$ must be obtained. In García et al. (2020), they propose a convex combination of the local zonotope with each received neighboring set. The weighting matrix for this combination can be obtained with an algebraic equation (AE). In Xia et al. (2018), they have to solve an optimization problem subject to a linear matrix inequality constraint (LMI) to obtain the new set, including the information of all neighbours at the same time. Finally, in

Alanwar et al. (2020), they propose an algebraic equation (AE) to find a vector of scalar weights for all the neighbours and the local agent.

Once solved the optimization problem, the *a posteriori* estimation set must be computed. In all the papers, this set is obtained as a linear combination of the estimation sets available for each agent. Hence, the computational cost of this step is similar for all the algorithms and, therefore, will not be considered in the analysis.

Table 1
Computational requirements

Algorithm	Computation	Method
Algorithm 1	$\sum_{\rho=1}^{\max\{\ell_i+1\}} \mathcal{O}(n_{i,\rho}, \sum_{j\in N_i} n_{j,\rho-1})$	AE
García et al. (2020)	$\bar{N} \times \mathcal{O}(n,n)$	AE
Xia et al. (2018)	$O(n,n)$	LMI
Alanwar et al. (2020)	$\mathcal{O}(1+\bar{N},1)$	AE

The computational requirements for obtaining matrix $N_{i,\rho}$ in (16) depends on the graph and output matrix of each agent, but because of the multi-hop subspace decomposition, it is, in general, more efficient than the algorithm in García et al. (2020), because the number of variables of each optimization problem is smaller (see Section 6 for a particular example). In addition, the method proposed here scales better with the number of neighbours and states.

6 Simulation and numerical results

6.1 System description

In order to verify the effectiveness and to show the implementation of the proposed observer, a simulation and numerical example is presented. The system is taken from del Nozal et al. (2017):

$$x(k+1) = \begin{bmatrix} 1.005 & 0 & 0 & 0 \\ 0 & 0.9954 & -0.08757 & 0 \\ 0 & 0.1248 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.9775 \end{bmatrix} x(k)+w(k).$$

A network of four agents is considered, each one measuring a different output y_i and communicating through a graph comprised of edges (1,2), (1,3), (2,4):

$$y_1 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} x + v_1,\ y_2 = \begin{bmatrix}0 & 0.5 & 0.5 & 0\end{bmatrix} x + v_2,$$
$$y_3 = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix} x + v_3,\ y_4 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix} x + v_4,$$

Local detectability is not achieved by any of the agents, but Assumption 2 is fulfilled. Noises and disturbances are bounded according to Assumption 1, with $R_i = 0.02, \forall i$, and $Q = 0,02I$.

6.2 Simulation results

The estimator proposed in Algorithm 1 will be compared to those in Alanwar et al. (2020), García et al. (2020), both distributed zonotopic observers. For the sake of a fairer comparison, the method in Alanwar et al. (2020) will be implemented assuming that no communication between the neighbours is allowed in the measurement update. In other words, we will compare here the benefit of the proposed observer based on the multi-hop decomposition, with the one based in intersections of neighboring sets García et al. (2020), and the diffusion algorithm Alanwar et al. (2020).

Table 2 includes the computational requirements for each method and every agent, illustrating the benefit of the proposed approach compared to that in García et al. (2020). Even, for agent 3, the proposed method is computationally more efficient than that in Alanwar et al. (2020).

Table 2
Computational requirements for the example

Alg.	Agent	Computation	Agent	Computation
Alg.1	1	$\mathcal{O}(2,4)+\mathcal{O}(1,2)$	3	$\mathcal{O}(1,1)+\mathcal{O}(1,1)$
	2	$\mathcal{O}(2,2)$	4	$\mathcal{O}(2,2)+\mathcal{O}(1,2)$
García et al. (2020)	1,2	$\mathcal{O}(4,4)+\mathcal{O}(4,4)$	3,4	$\mathcal{O}(4,4)$
Alanwar et al. (2020)	1,2	$\mathcal{O}(3,1)$	3,4	$\mathcal{O}(2,1)$

In order to present a numerical comparison of the performance of the observers, we define the next index for a time horizon of N steps and agent i:

$$I_i = \frac{1}{N}\sum_{k=1}^{N}\sqrt{\sum_{\rho=0}^{\ell_{i+1}} ||\hat{\mathcal{Z}}_{i,\rho}(k|k)||_F^2},$$

where, with some abuse of notation, $\ell_i = 0, \forall i$, for the methods in Alanwar et al. (2020), García et al. (2020), in which only one hop is computed. This index tries to capture the average value of the Frobenius norm of the sets computed for all the hops and time instants. The initial condition for the simulation is $x(0) = [0, 0.5, 0.5, 0.9]$. Noises and disturbances are generated randomly meeting Assumption 1, but the same random sequence is applied to the three algorithms. For the sake of a fair comparison, the same reduction order algorithm is implemented in all cases (see Section 2).

Table 3 shows the value of the aforementioned indexes for different values of the order of the zonotope. It also includes factor $\mathcal{T}$, which measures the computational time increment [1] with respect to the case $q = 10$. That is, $\mathcal{T}$

[1] Authors have intentionally avoided to include the computational time for each algorithm, because the implementation (efficient coding of the instructions) of the listed algorithms might influence this absolute value.

measures how the computational time increases as the order of the zonotopes q grows (in Section 6.3, we will analyse the increment of the computational time as the dimension of the systems grows using this same factor). It is important to remark that, in a practical implementation, all the estimators run in parallel. However, this factor has been computed by making sequential and non-parallelized simulations for each agent using an unique computer (Intel Core i5-3320M at 2.6GHz, 8Gb of RAM, Windows 10).

Table 3
Numerical results

q	Algorithm	Index	$\mathcal{T}$
10	Alg. 1	$I = [0.0818, 0.1689, 0.1447, 0.2108]$	1
	García et al. (2020)	$I = [0.1421, 0.1774, 0.1647, 0.1976]$	1
	Alanwar et al. (2020)	$I = [413.26, 2.4050, 1.4753, 51.499]$	1
	Centralized	$I = 0.0521$	1
100	Alg. 1	$I = [0.0535, 0.0821, 0.0838, 0.0814]$	1,32
	García et al. (2020)	$I = [0.0790, 0.0797, 0.0826, 0.0759]$	1,37
	Alanwar et al. (2020)	$I = [27.323, 0.5470, 0.4458, 2.3266]$	1,59
	Centralized	$I = 0.0379$	1,41
500	Alg. 1	$I = [0.0380, 0.0821, 0.0830, 0.0567]$	2,44
	García et al. (2020)	$I = [0.0623, 0.0592, 0.0677, 0.0599]$	6,01
	Alanwar et al. (2020)	$I = [5.1630, 0.4363, 0.2953, 1.3510]$	6,80
	Centralized	$I = 0.0379$	4,01

Table 3 illustrates that the smaller the order becomes, the worst performance is obtained in all the approaches. However, the methods in Alanwar et al. (2020), García et al. (2020) are more sensitive than the one proposed in this paper, and their performance decay faster when the order decreases. On the other hand, the greater the order is, the higher the computational times. Again, the proposed method presents a better decay rate in what respect to computational times.

The diffusion method in Alanwar et al. (2020) obtains very poor responses, specially when estimating the oscillatory modes by agents that do not directly measured them. It can be concluded that the diffusion algorithm, just by itself, is not able to produce nice estimations. It's worth reminding that the authors in Alanwar et al. (2020) proposed to share the measurements between neighbours. Of course, this will increase the performance of the estimation.

The case of $q = 100$ helps us understanding an interesting feature of the proposed method based on the multi-hop subspace decomposition. Agent 1 makes use of the information measured by all the agents, since agents 2 and 3, the ones measuring the oscillatory modes, are located at hop $\rho = 1$, and agent 4 is located at hop $\rho = 2$.

However, agent 2 is able to estimate the whole oscillatory modes only with the local measurement, so it will not receive any information incoming from agent 3, through agent 1. This means that the proposed observer, pursuing a reduction of the complexity, does not exploit all

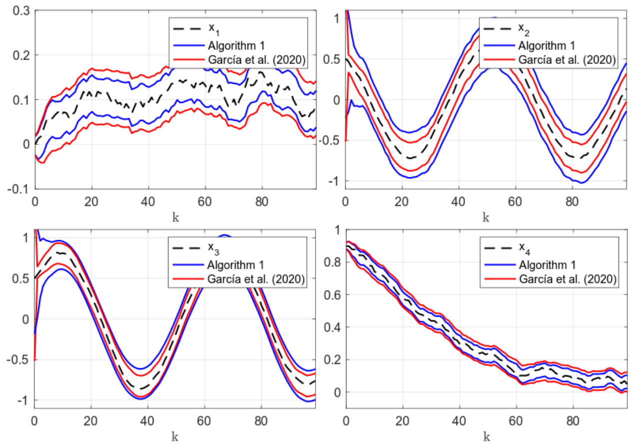

Fig. 1. Estimation performance by agent 2.

the information available in the graph. This is the reason why the performance of agent 2 is a bit slower than that achieved by the algorithm in García et al. (2020). This fact opens us the door for future extensions of the work. Note that when all the available information is used, as for agent 1, the proposed method yields better performance (see Table 3) with reduced computational requirements (see Table 2), and almost matches the one obtained with the centralized algorithm.

Finally, for a maximum order of $q = 100$, Figure 1 depict some simulation results for the proposed algorithm and the one in García et al. (2020). As Table 3 predicted, the estimation of oscillating modes by agent 2 using the method in García et al. (2020) is slightly better than that obtained with the method proposed in this paper. Exploiting the redundant information is left for future extensions.

6.3 *Scalability analysis*

Consider now that the dynamic matrix of the system changes, so that block (1,1), this is $A(1,1) = 1.005$, is now changed to:

$$A^*(1,1) = \begin{bmatrix} 1.005 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix},$$

scaling the number of states of the system from 4 to 6. The output matrices for the four agents are accordingly re-scaled:

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; C_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Table 4 analyses the scalability of the methods through the new time factor $\mathcal{T}^*$, that measures the ratio between the computational time needed for the enlarged system and the time for the original system with 4 states. The same graph, noises and disturbances are considered. Again, the proposed method outperforms others in the literature in terms of computational scalability.

Table 4
Time Factor $\mathcal{T}^*$ for the scalability analysis

Algorithm	Alg. 1	García et al. (2020)	Alanwar et al. (2020)	Centralized
q=100	1.08	1.65	1.89	1.11
q=500	2.45	2.60	2.98	2.50

7 Conclusions

A distributed set-membership estimator for linear full-coupled systems affected by bounded disturbances has been presented in this paper. The multi-hop decomposition has allowed to propose a simple design the observers to minimize the *F-radius* of the estimation sets.

Simulations have shown that the performance is comparable with published results that demand higher computational resources, and approaches the performance of the centralized case when all information is exploited. In addition, it can be concluded that the proposed method is more robust, in terms of performance and computational time increments, to reductions of the maximum order of the zonotopes. Furthermore, it computationally scales better than others when the dimension of the system grows.

Future work will include the design and performance analysis of a distributed set-membership observer that takes into account redundant information.

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References

Alanwar, A., Rath, J., Said, H. & Althoff, M. (2020), 'Distributed Set-Based Observers Using Diffusion Strategy', *arXiv preprint arXiv:2003.10347* .

Battistelli, G. & Chisci, L. (2014), 'Kullback–Leibler average, consensus on probability densities, and distributed state estimation with guaranteed stability', *Automatica* **50**(3), 707–718.

Chen, B., Hu, G., Ho, D. & Yu, L. (2019), 'Distributed Kalman filtering for time-varying discrete sequential systems', *Automatica* **99**, 228 – 236.

Chen, T., Foo, Y., Ling, K. & Chen, X. (2017), 'Distributed State Estimation Using a Modified Partitioned Moving Horizon Strategy for Power Systems', *Sensors* **17**(10), 2310.

Combastel, C. (2015), 'Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence', *Automatica* **55**(1), 265–273.

Combastel, C. & Zolghadri, A. (2018), 'FDI in cyber physical systems: A distributed zonotopic and gaussian Kalman filter with bit-level reduction', *IFAC-PapersOnLine* **51**(24), 776–783.

del Nozal, A., Orihuela, L. & Millán, P. (2017), 'Distributed consensus-based Kalman filtering considering subspace decomposition', *IFAC-PapersOnLine* **50**(1), 2494–2499.

del Nozal, A. R., Millán, P., Orihuela, L., Seuret, A. & Zaccarian, L. (2019), 'Distributed estimation based on multi-hop subspace decomposition', *Automatica* **99**, 213–220.

Farina, M., Ferrari-Trecate, G. & Scattolini, R. (2010), 'Distributed moving horizon estimation for linear constrained systems', *IEEE Transactions on Automatic Control* **55**(11), 2462–2475.

García, R. A., Orihuela, L., Millán, P., Rubio, F. R. & Ortega, M. G. (2020), 'Guaranteed estimation and distributed control of vehicle formations', *International Journal of Control* **0**(0), 1–14.

Ge, X., Han, Q. & Wang, Z. (2019), 'A Threshold-Parameter-Dependent Approach to Designing Distributed Event-Triggered H_∞ Consensus Filters Over Sensor Networks', *IEEE Transactions on Cybernetics* **49**(4), 1148–1159.

Ge, X., Han, Q., Zhang, X., Ding, L. & Yang, F. (2019), 'Distributed Event-Triggered Estimation Over Sensor Networks: A Survey', *IEEE Transactions on Cybernetics* **Online**, 1–15.

He, X., Hu, C., Hong, Y., Shi, L. & Fang, H. (2020), 'Distributed Kalman Filters with State Equality Constraints: Time-Based and Event-Triggered Communications', *IEEE Transactions on Automatic Control* **65**(1), 28–43.

Ierardi, C., Orihuela, L. & Jurado, I. (2019), 'Distributed Estimation Techniques for Cyber-Physical Systems: A Systematic Review', *Sensors* **19**(21), 4720.

Kharkovskaia, T., Efimov, D., Fridman, E., Polyakov, A. & Richard, J.-P. (2020), 'Interval observer design and control of uncertain non-homogeneous heat equations', *Automatica* **111**, 108595.

Kopetzki, A. K., Schürmann, B. & Althoff, M. (2017), Methods for order reduction of zonotopes, *in* '2017 IEEE 56th Annual Conference on Decision and Control', IEEE, pp. 5626–5633.

Liu, S., Wang, Z., Wei, G. & Li, M. (2020), 'Distributed Set-Membership Filtering for Multirate Systems Under the Round-Robin Scheduling Over Sensor Networks', *IEEE Transactions on Cybernetics* **50**(5), 1910–1920.

Ma, L., Wang, Z., Lam, H. & Kyriakoulis, N. (2016), 'Distributed event-based set-membership filtering for a class of nonlinear systems with sensor saturations over sensor networks', *IEEE transactions on cybernetics* **47**(11), 3772–3783.

Mitra, A. & Sundaram, S. (2018), 'Distributed observers for LTI systems', *IEEE Transactions on Automatic Control* **63**(11), 3689–3704.

Orihuela, L., Ierardi, C. & Jurado, I. (2020), A distributed set-membership estimator for linear systems considering multi-hop subspace decomposition, *in* 'IFAC World Congress', IFAC.

Orihuela, L., Millán, P., Roshany-Yamchi, S. & García, R. A. (2018), 'Negotiated distributed estimation with guaranteed performance for bandwidth-limited situations', *Automatica* **87**, 94–102.

Orihuela, L., Roshany-Yamchi, S., García, R. & Millán, P. (2018), 'Distributed set-membership observers for interconnected multi-rate systems', *Automatica* **85**, 221–226.

Tang, W., Wang, Z., Wang, Y., Raïssi, T. & Shen, Y. (2019), 'Interval estimation methods for discrete-time linear time-invariant systems', *IEEE Transactions on Automatic Control* **64**(11), 4717–4724.

Ugrinovskii, V. & Fridman, E. (2014), 'A Round-Robin type protocol for distributed estimation with H_∞ consensus', *Systems & Control Letters* **69**, 103 – 110.

Wang, Y., Alamo, T., Puig, V. & Cembrano, G. (2017), 'Distributed zonotopic set-membership state estimation based on optimization methods with partial projection', *IFAC-PapersOnLine* **50**(1), 4039 – 4044. 20th IFAC World Congress.

Wang, Y., Alamo, T., Puig, V. & Cembrano, G. (2018), A Distributed Set-membership Approach based on Zonotopes for Interconnected Systems, *in* '2018 IEEE Conference on Decision and Control', pp. 668–673.

Xia, N., Yang, F. & Han, Q. (2018), 'Distributed networked set-membership filtering with ellipsoidal state estimations', *Information Sciences* **432**, 52–62.

Yin, X. & Liu, J. (2017), 'Distributed moving horizon state estimation of two-time-scale nonlinear systems', *Automatica* **79**, 152–161.

Zhang, Y., Wang, J. & Li, Z. (2020), 'Interval State Estimation with Uncertainty of Distributed Generation and Line Parameters in Unbalanced Distribution Systems', *IEEE Transactions on Power Systems* **35**(1), 762–772.

Zou, L., Wang, Z., Hu, J. & Han, Q.-L. (2020), 'Moving horizon estimation meets multi-sensor information fusion: Development, opportunities and challenges', *Information Fusion* **60**, 1 – 10.