



Zeros of polynomials orthogonal with respect to a signed weight

M.J. Atia^{a,*}, M. Benabdallah^b, R.S. Costas-Santos^c

^a *Faculté des sciences de Gabès, université de Gabès, Tunisie*

^b *Lycée Blidet, Blidet Kebili, Tunisie*

^c *Department of Mathematics, University of California, South Hall, Room 6607 Santa Barbara, CA 93106, USA*

Received 7 May 2010; received in revised form 30 August 2011; accepted 20 September 2011

Communicated by F. Beukers

Abstract

In this paper we consider the monic polynomial sequence $(P_n^{\alpha,q}(x))$ that is orthogonal on $[-1, 1]$ with respect to the weight function $x^{2q+1}(1-x^2)^\alpha(1-x)$, $\alpha > -1$, $q \in \mathbb{N}$; we obtain the coefficients of the tree-term recurrence relation (TTRR) by using a different method from the one derived in Atia et al. (2002) [2]; we prove that the interlacing property does not hold properly for $(P_n^{\alpha,q}(x))$; and we also prove that, if $x_{n,n}^{\alpha+i,q+j}$ is the largest zero of $P_n^{\alpha+i,q+j}(x)$, $x_{2n-2j,2n-2j}^{\alpha+j,q+j} < x_{2n-2i,2n-2i}^{\alpha+i,q+i}$, $0 \leq i < j \leq n-1$. Crown Copyright © 2011 Published by Elsevier B.V. on behalf of Royal Netherlands Academy of Arts and Sciences. All rights reserved.

MSC: primary 33C05; 33C20; 42C05; 30C15

Keywords: Zeros; Real-rooted polynomials; Generalized Jacobi polynomials; Generalized Gegenbauer polynomials

Contents

1. Introduction.....	27
2. Basic definitions and preliminary results	27

* Corresponding author.

E-mail addresses: jalel.atia@gmail.com (M.J. Atia), benabdallahmajed@gmail.com (M. Benabdallah), rscosa@gmail.com (R.S. Costas-Santos).

URL: <http://www.rscosan.com> (R.S. Costas-Santos).

0019-3577/\$ - see front matter Crown Copyright © 2011 Published by Elsevier B.V. on behalf of Royal Netherlands Academy of Arts and Sciences. All rights reserved.

doi:10.1016/j.indag.2011.09.011

3. Algebraic relations between $(P_n^{\alpha,q}(x))$ and the GG-polynomials	28
4. Zeros of $(P_n^{\alpha,q})$	29
Acknowledgments	30
References	30

1. Introduction

It is a well known fact that if (p_n) is orthogonal with respect to a (real) weight function, namely $w(x)$, and such weight function is positive on $[a, b]$, then the zeros of p_n are real, distinct, interlace, and lie inside $]a, b[$, but such an interlacing property is no longer valid when the weight is a signed function. In fact, Perron [9] proved that when the $w(x)$ changes sign once then one of zeros can lie outside of $[a, b]$.

In this paper we prove that such zero can lie into one of the endpoints of the interval, a or b . We consider the monic polynomial sequence $(P_n^{\alpha,q}(x))$ that is orthogonal on $[-1, 1]$ with respect to the weight function $x^{2q+1}(1-x^2)^\alpha(1-x)$, $\alpha > -1$, $q \in \mathbb{N}$ that changes sign once, at $x = 0$, and we prove that all the zeros are real, non interlacing, and that one of the zeros is the endpoint $a = -1$.

The sequence of monic orthogonal polynomials $(P_n^{\alpha,q}(x))$ satisfies for $n \geq 0$ the following TTRR [4]:

$$P_{n+2}^{\alpha,q}(x) = (x - \beta_{n+1}^{\alpha,q})P_{n+1}^{\alpha,q}(x) - \gamma_{n+1}^{\alpha,q}P_n^{\alpha,q}(x), \tag{1}$$

with initial conditions $P_0^{\alpha,q}(x) = 1$, $P_1^{\alpha,q}(x) = x - \beta_0^{\alpha,q}$, being $(\beta_n^{\alpha,q})$ and $(\gamma_n^{\alpha,q})$ the coefficients of the recurrence relation. They were calculated in [2] by using the Laguerre–Freud equations, and, later on, an explicit expression for $P_n^{\alpha,q}(x)$ was given in [1]. The main aim of this paper is to keep studying these polynomials, more precisely, the behavior of the zeros of $(P_n^{\alpha,q}(x))$.

People working on zeros of orthogonal polynomials know how difficult it is to explore this area. In fact, even in the case of Jacobi polynomials results on zeros are presented as conjectures (see [5,6]).

In order to do this study, we use generalized Gegenbauer polynomials $(GG_n^{\alpha,\mu})$ that are orthogonal on $[-1, 1]$ with respect to the weight function $|x|^\mu(1-x^2)^\alpha$, $\alpha > -1$, $\mu > -1$. Actually, $GG_{2n}^{\alpha,\mu}(x) = J_n^{\alpha, \frac{\mu-1}{2}}(x^2)$ ($GG_n^{\alpha,\mu}(x) = x^{\frac{1-(-1)^n}{2}} J_{[\frac{n}{2}] }^{\alpha, \frac{\mu-(-1)^n}{2}}(x^2)$) where the J_n are the Jacobi polynomials on the interval $[0, 1]$ with weight $x^{\frac{\mu-1}{2}}(1-x)^\alpha$ and are given by the classical formula

$$J_n^{\alpha, \frac{\mu-1}{2}}(x) = x^{-\frac{\mu-1}{2}}(1-x)^{-\alpha} \left(\frac{d}{dx} \right)^n (x^{n+\frac{\mu-1}{2}}(1-x)^{n+\alpha}).$$

Some properties of GG-polynomials can be found in [10,3].

The structure of the paper is the following: in Section 2 we present basic definitions, some notations, and a few preliminary results, in Section 3 we obtain some algebraic relations between $(P_n^{\alpha,q}(x))$ and the GG-polynomials as well as the recurrence coefficients of the TTRR fulfilled by $(P_n^{\alpha,q}(x))$, and in Section 4 some results regarding zeros of $(P_n^{\alpha,q}(x))$ are given.

2. Basic definitions and preliminary results

The *Pochhammer symbol*, or shifted factorial, is defined as

$$(\alpha)_0 = 1, \quad (\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad n \geq 1. \tag{2}$$

The Gauss's hypergeometric function is

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n, \quad \alpha, \beta \in \mathbb{C}; \gamma \in \mathbb{C} \setminus \mathbb{Z}_-; |z| < 1. \tag{3}$$

When α , or β , is a negative integer the hypergeometric series (3) terminates, i.e. it reduces to a polynomial (of degree α , or β resp.).

Observe that after straightforward calculation one gets

$${}_2F_1(-m, \beta; \gamma; z) = \sum_{n=0}^m \frac{(-m)_n(\beta)_n}{n!(\gamma)_n} z^n = \sum_{n=0}^m \binom{m}{n} \frac{(\beta)_n}{(\gamma)_n} (-z)^n. \tag{4}$$

Denoting by

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) &\equiv F, & {}_2F_1(\alpha \pm 1, \beta; \gamma; z) &\equiv F(\alpha \pm 1), \\ {}_2F_1(\alpha, \beta \pm 1; \gamma; z) &\equiv F(\beta \pm 1), & {}_2F_1(\alpha, \beta; \gamma \pm 1; z) &\equiv F(\gamma \pm 1). \end{aligned}$$

The functions $F(\alpha \pm 1)$, $F(\beta \pm 1)$, and $F(\gamma \pm 1)$ are said to be *contiguous* of F [7, p. 242]. Among the relations of this type we cite the following ones:

$$(\alpha - \beta)F - \alpha F(\alpha + 1) + \beta F(\beta + 1) = 0, \tag{5}$$

$$(\alpha - \beta)(1 - z)F + (\gamma - \alpha)F(\alpha - 1) - (\gamma - \beta)F(\beta - 1) = 0. \tag{6}$$

Definition 2.1. For $n \geq 0$, GG polynomials are given by

$$GG_{2n}^{\alpha, \mu}(x) = x^{2n} {}_2F_1\left(-n, -n - \frac{\mu}{2} + \frac{1}{2}; -2n - \alpha - \frac{\mu}{2} + \frac{1}{2}; \frac{1}{x^2}\right) \tag{7}$$

$$GG_{2n+1}^{\alpha, \mu}(x) = x^{2n+1} {}_2F_1\left(-n, -n - \frac{\mu}{2} - \frac{1}{2}; -2n - \alpha - \frac{\mu}{2} - \frac{1}{2}; \frac{1}{x^2}\right). \tag{8}$$

3. Algebraic relations between $(P_n^{\alpha, q}(x))$ and the GG-polynomials

Proposition 3.1. For any $n \geq 0$, and any integer q the following identities hold:

$$P_{2n}^{\alpha, q}(x) = GG_{2n}^{\alpha, 2q+2}(x), \tag{9}$$

$$P_{2n+1}^{\alpha, q}(x) = (1 + x)GG_{2n}^{\alpha+1, 2q+2}(x). \tag{10}$$

Remark 3.1. Observe that with (9), we can write the last equation as

$$P_{2n+1}^{\alpha, q}(x) = (1 + x)P_{2n}^{\alpha+1, q}(x), \tag{11}$$

one should point out that we have α in the left hand side whereas we have $\alpha + 1$ in the right hand side.

Proof. One can easily show that

$$\int_{-1}^1 x^{2q+1} (1 - x^2)^\alpha (1 - x)x^k GG_{2n}^{\alpha, 2q+2}(x) dx = 0, \quad 0 \leq k \leq 2n - 1.$$

They all follow from the orthogonality property of GG_{2n} , except for $k = 0$, where the oddity of x^{2q+1} is used in addition. It can also be shown that

$$\int_{-1}^1 x^{2q+1}(1-x^2)^\alpha(1-x)x^k(1+x)GG_{2n}^{\alpha+1,2q+2}(x)dx = 0, \quad 0 \leq k \leq 2n.$$

They all follow from the orthogonality property of GG_{2n} , except for $k = 2n$, where again the oddity of x^{2q+1} is important.

These identities immediately imply that the polynomials

$$P_{2n}^{\alpha,q} = GG_{2n}^{\alpha,2q+2}, \quad P_{2n+1}^{\alpha,q} = (1+x)GG_{2n}^{\alpha+1,2q+2}$$

from a sequence of polynomials, orthogonal with respect to the weight $x^{2q+1}(1-x)^2(1-x)$. \square

Remark 3.2. By using Eq. (7) we get for $n \geq 0$ the hypergeometric representation for $P_{2n}^{\alpha,q}(x)$:

$$P_{2n}^{\alpha,q}(x) = x^{2n} {}_2F_1\left(-n, -n - q - \frac{1}{2}; -2n - \alpha - q - \frac{1}{2}; \frac{1}{x^2}\right), \tag{12}$$

that we can be written as [8, V1 p. 40 (23)]

$$P_{2n}^{\alpha,q}(x) = \frac{(-1)^n \left(q + \frac{3}{2}\right)_n}{\left(n + q + \alpha + \frac{3}{2}\right)_n} {}_2F_1\left(-n, n + q + \alpha + \frac{3}{2}; q + \frac{3}{2}; x^2\right). \tag{13}$$

Once we have got these algebraic relations we can compute the recurrences coefficients associated to the polynomial sequence $(P_n^{\alpha,q}(x))$.

Remark 3.3. Notice that due the expression of the integrals and the weight functions we can find a link between the polynomials $(P_n^{\alpha,q})$ and GG -polynomials, which was not possible to do with Laguerre–Freud equation [2] or with the explicit representation of $P_n^{\alpha,q}(x)$ [1].

Proposition 3.2. *The recurrence coefficients of the monic polynomial sequence $(P_n^{\alpha,q}(x))$ fulfills (1) are given by*

$$\beta_n^{\alpha,q} = (-1)^{n+1}, \quad n \geq 0 \tag{14}$$

$$\gamma_{2n}^{\alpha,q} = -2 \frac{n(2n + 2q + 1)}{(4n + 2\alpha + 2q + 1)(4n + 2\alpha + 2q + 3)}, \quad n \geq 1 \tag{15}$$

$$\gamma_{2n+1}^{\alpha,q} = -2 \frac{(n + \alpha + 1)(2n + 2\alpha + 2q + 3)}{(4n + 2\alpha + 2q + 3)(4n + 2\alpha + 2q + 5)}, \quad n \geq 0. \tag{16}$$

Proof. These recurrence coefficients follow from the contiguity relations between hypergeometric functions. \square

4. Zeros of $(P_n^{\alpha,q})$

Using (9) and (10) we can state the following result:

Theorem 4.1. *The following statements hold:*

- All the zeros of $P_n^{\alpha,q}(x)$ are real.
- The Perron's zero is -1 .
- The zeros of $P_{2n}^{\alpha,q}(x)$ and the zeros of $P_{2n+1}^{\alpha,q}(x)$ do not interlace.

Proof. By (9) and (10), the first two statements follow. To prove the third one, it is sufficient to point out that the zeros of $GG_{2n}^{\alpha,\mu}$ and $GG_{2n}^{\alpha+1,\mu}$ are non-interlacing. However, both polynomials are even of the same degree and non-zero at the origin. It is straightforward that two such polynomials cannot have interlacing zero sets. The zeros do not coincide because the parameter α becomes $\alpha + 1$ and μ is unchanged. \square

Proposition 4.2. *If we denote by $x_{n,n}^{\alpha+i,q+j}$ the largest zero of $P_n^{\alpha+i,q+j}$, then*

$$x_{2n-2j,2n-2j}^{\alpha+j,q+j} < x_{2n-2i,2n-2i}^{\alpha+i,q+i}, \quad 0 \leq i < j \leq n-1. \quad (17)$$

Proof. $P_{2n}^{\alpha,q}(x) = GG_{2n}^{\alpha,2q+2}(x) = J_n^{\alpha, \frac{2q+1}{2}}(x^2)$, then zeros of $J_n^{\alpha, \frac{2q+1}{2}}$ are the same as those of $K_n^{\alpha, \frac{2q+1}{2}}$ with $K_n^{\alpha, \frac{2q+1}{2}}(x) = \left(\frac{d}{dx}\right)^n (x^{n+\frac{2q+1}{2}}(1-x)^{\alpha+n})$ thus

$$K_n^{\alpha, \frac{2q+1}{2}}(x) = \frac{d}{dx} K_{n-1}^{\alpha+1, \frac{2q+3}{2}}(x)$$

this implies that between two consecutive zeros of $K_n^{\alpha, \frac{2q+1}{2}}$ there exists one zero of $K_{n-1}^{\alpha+1, \frac{2q+3}{2}}$ and then the largest zero of $P_{2n}^{\alpha,q}$ is greater than $P_{2(n-1)}^{\alpha+1,q+1}$ and so on for the largest zero of $P_{2(n-1)}^{\alpha+1,q+1}$, $P_{2(n-2)}^{\alpha+2,q+2}, \dots$ \square

Remark 4.1. Using Proposition 4.2. and the relation

$$x_{2n+1,m}^{\alpha+k,q+l} = x_{2n,m-1}^{\alpha+k+1,q+l}; \quad k, l \in \mathbb{N}, 2 \leq m \leq 2n+1,$$

with $x_{2n+1,1}^{\alpha+k,q+l} = -1$, we obtain

$$x_{2n-2j+1,2n-2j+1}^{\alpha+j-1,q+j} < x_{2n-2i+1,2n-2i+1}^{\alpha+i-1,q+i}, \quad 0 \leq i < j \leq n.$$

Acknowledgments

The authors want to thank Prof. F. Marcellan and Prof. H. Stahl for their discussions and remarks which helped us to improve the representation of this paper. Special thanks to the referee for valuable comments, suggestions and remarks.

RSCS acknowledges financial support from the Ministerio de Ciencia e Innovación of Spain and under the program of postdoctoral grants (Programa de becas postdoctorales) and grant MTM2009-12740-C03-01.

References

- [1] M.J. Atia, Explicit representations of some orthogonal polynomials, *Integral Transforms Spec. Funct.* 18 (10) (2007) 731–742.
- [2] M.J. Atia, F. Marcellan, I.A. Rocha, On semi-classical orthogonal polynomials: a quasi-definite functional of class 1, *Facta Universitatis, Nis. Ser. Math. Inform.* 17 (2002) 25–46.

- [3] M. Benabdallah, M.J. Atia, Rodrigues type formula for some generalized Jacobi, *J. Math. Syst. Sci.* (in press).
- [4] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [5] W. Gautschi, On a conjectured inequality for the largest zero of Jacobi polynomials, *Numer. Algor.* 49 (1–4) (2008) 195–198.
- [6] W. Gautschi, On conjectured inequalities for zeros of Jacobi polynomials, *Numer. Algor.* 50 (2009) 93–96.
- [7] L.L. Lebedev, *Special Functions and Their Applications*, Dover Publications, Inc., New York, 1972.
- [8] Y.L. Luke, *The Special Functions and Their Approximations*, in: *Mathematics in Science & Engineering*, vol. 1, Academic Press Inc., 1969.
- [9] O. Perron, *Die Lehre Von Den Kettenbrüchen*, Teubner, Berlin, 1929.
- [10] G. Szegő, *Orthogonal polynomials*, in: *American Mathematical Society Colloquium Publications*, vol. 23, American Mathematical Society, Providence, RI, 1939.